NONEQUILIBRIUM DENSITY FLUCTUATIONS FOR THE ZERO RANGE PROCESS WITH COLOUR

HANNA K. JANKOWSKI

ABSTRACT. We examine the fluctuations of the empirical density measure for the colour version of the symmetric nearest neighbour zero range particle systems in dimension one. We show that the weak limit of these fluctuations is the solution of a system of coupled generalized Ornstein-Uhlenbeck processes. We also discuss how this result may be used to prove a central limit theorem for the tagged particle on the level of finite dimensional distributions, and identify the limiting variance. This is the central limit theorem associated to propagation of chaos for this interacting particle system.

1. INTRODUCTION

The zero range particle system describes a class of microscopic models. It was first introduced by Spitzer in 1970 [Spi] as an example of an interacting particle system, and has since been studied at length. One of its greatest advantages is its mathematical tractability. It has found widespread application in the modelling of nonequilibrium phenomena. Examples of these include models of sandpile dynamics and other flow mechanisms, as well as the repton model of gel electrophoresis [Eva].

In the zero range system we consider particles which begin from a random configuration and move around the discrete circle $\mathbb{Z}/N\mathbb{Z}$ following these dynamics: each particle waits an exponential amount of time to jump, it then chooses one of its nearest neighbours with equal probability. The exponential rate for the first particle to leave a particular site is a function $c(\cdot)$ of the number of particles at this site, and it is for this reason that the system is called the zero range model. Configurations of the system are denoted by η; if we are currently at site $x \in \mathbb{Z}/N\mathbb{Z}$ then the number of particles at the site is denoted by $\eta(x)$. Next, we differentiate between particles by assigning to each one of k colours. The dynamics for each particle are the same as previously, however, in the colour process we keep track of the number of particles of each colour at all sites x of the discrete circle, and we denote this as $\eta_i(x)$. If we ignore the colour of each particle then we obtain the colour-blind model, which simply keeps track of the total number of particles at each site x. We will say that the colour version *contracts* to the colour-blind process.

The dynamics we have described conserve the total number of particles of each colour. Furthermore, the configurations have a family of invariant measures, indexed

by the average density vector $\boldsymbol{\rho}$. These are usually referred to as the grand canonical measures, and we denote them here by $\mu_{\boldsymbol{\rho}}$. For the colour-blind model these contract to the measures μ_ρ indexed by the total particle density.

The above construction creates the model on a microscopic level. We, however, are interested in the system from a macroscopic viewpoint. In order to achieve this we re-scale space by $\frac{1}{N}$. We then need to speed up time by N^2 so as to obtain a non-trivial system evolution. This is the standard diffusive space-time re-scaling. We will denote the configurations of the re-scaled system by η^N and η_i^N for the colour-blind and k-colour systems respectively. After re-scaling, the particles move around the discrete subset $\mathbb{T}^N = \{0, \ldots, (N-1)/N\}$ of the unit circle $\mathbb{T}$.

We first wish to understand the evolution of the macroscopic densities for the colour process. To do this, we study the asymptotic behaviour of the empirical densities

$$\Pi_{i,t}^N = \tfrac{1}{N} \sum_{x \in \mathbb{T}^N} \eta_{i,t}^N(x).$$

We assume that the system is started in such a way so that the initial empirical densities correspond to some fixed macroscopic densities

$$\lim_{N \to \infty} \Pi_{i,0}^N = \rho_0^i(x)dx, \tag{1.1}$$

in the sense that this limit exists weakly on $\mathbb{T}$.

In the colour-blind model it is well known that (see for example [KL]) that the macroscopic densities $\rho_t(x)$ evolve according to the equation

$$\partial_t \rho = \tfrac{1}{2}\nabla D(\rho)\nabla \rho, \quad \rho_t|_{t=0} = \rho_0 \tag{1.2}$$

with $\rho_0 = \sum \rho_0^i$ given above. We call $D(\rho)$ the *bulk* diffusion coefficient. Results of this type are known as hydrodynamic scaling limits in the literature.

It is also known that the empirical measures $\Pi_{i,t}^N$ satisfy a law of large numbers and converge weakly to a non-random limit $\rho_t^i(x)dx$ (see [GJLL] for a discussion). The vector $\boldsymbol{\rho}_t$, describing the limit, is the unique solution of the coupled non-linear differential equation

$$\partial_t \boldsymbol{\rho} = \frac{1}{2}\nabla \boldsymbol{D}_k(\boldsymbol{\rho})\nabla \boldsymbol{\rho}, \tag{1.3}$$

with boundary condition $\boldsymbol{\rho}_0 = \{\rho_0^i\}_{i=1}^k$. Here $\boldsymbol{D}_k(\boldsymbol{\rho})$ is the colour diffusion coefficient and is given by the formula

$$\boldsymbol{D}_k(\boldsymbol{\rho}) = S(\rho)I + \frac{(D-S)(\rho)}{\rho} \begin{bmatrix} \rho^1 & \cdots & \rho^1 \\ \vdots & & \vdots \\ \rho^k & \cdots & \rho^k \end{bmatrix}, \tag{1.4}$$

where I is the $k \times k$ identity matrix and $S(\rho)$ is the *self* diffusion coefficient which we now describe.

Consider the zero range particle system where we have singled out, or *tagged*, one of the particles. The remaining particles are combined together to define a random environment of the tagged particle. This new system is a Markov process with identifiable

invariant measures. Using the methods of [KV] it is straightforward to show that this tagged particle started in equilibrium will converge to a diffusion with generator

$$\mathcal{L} = \frac{1}{2}S(\rho)\triangle.$$

$S(\rho)$ is called the self diffusion coefficient. One of the nice features of the zero range model is that we can calculate $S(\rho)$ explicitly. Additionally, in the zero range case, we may express the difference between the bulk and self diffusion coefficients, $D - S$, as $S'(\rho)\rho$. We calculate these coefficients explicitly for the zero range model in Remark 3.1. We also use the notation $\chi(\rho)$ to denote the static compressibility. For the zero range model this is the same as the variance of the variable $\eta(x)$ under the invariant measure μ_ρ, and the following identity links the static compressibility with the bulk and self diffusion coefficients:

$$S(\rho)\,\rho = \chi(\rho) \cdot D(\rho).$$

It is important to note that this is a particular property which holds for the zero range process, but not for general systems.

The hydrodynamic scaling limit for the colour densities of an interacting particle system was first studied in [Qua] for the symmetric simple exclusion process. This is a more difficult problem as the simple exclusion colour process is of *non-gradient* type.

In this paper we study the central limit theorem associated to the law of large numbers described above. That is, we consider the fields defined by

$$< Y_{i,t}^N, f > = \sqrt{N}\left\{\frac{1}{N}\sum_{x\in\mathbb{T}_N} f(x)[\eta_{i,t}^N(x) - \rho_t^i(x)]\right\}. \tag{1.5}$$

The main result is the following theorem, which identifies the asymptotic behaviour of $\boldsymbol{Y}_t^N$. Let $\boldsymbol{\rho}_t$ be the solution of (1.3). Denote by P_N the measure on $D\{[0,T], H_{-4}^k\}$ induced by the stochastic field $\boldsymbol{Y}^N$.

Theorem 1.1. *Under the assumptions given in Section 3.3, P_N converges weakly in $D\{[0,T], H_{-4}^2\}$ to a coupled generalized Ornstein-Uhlenbeck process characterised by the coupled generalized stochastic differential equation*

$$d\boldsymbol{Y}_t = \frac{1}{2}\{\triangle\boldsymbol{D}_k(\boldsymbol{\rho}_t)\}\boldsymbol{Y}_t dt + \{\nabla\boldsymbol{A}_k^{1/2}(\boldsymbol{\rho}_t)\}d\boldsymbol{W}_t \tag{1.6}$$

with initial condition $\boldsymbol{Y}_0 \sim P_0$. In the above $\boldsymbol{D}_k$ is the colour diffusion coefficient described previously, $\boldsymbol{A}_k$ is given by

$$\boldsymbol{A}_k(\boldsymbol{\rho}) = \frac{[\chi D](\rho)}{\rho}\,diag\{\rho^1, \ldots, \rho_k\}, \tag{1.7}$$

where $\chi(\rho)$ is the static compressibility, and D is again the bulk diffusion coefficient. $\boldsymbol{W}_t$ is the k-dimensional Gaussian random field defined as a vector composed of k

independent copies of the Gaussian random field W_t^i, with covariance

$$E[\langle W_t^i, f \rangle \langle W_s^i, g \rangle] = \min(s,t) < f, g >,$$

for $i = 1, \ldots, k$.

The emergence of the colour diffusion coefficient in (1.6) is not surprising, as the equation may be obtained by formally linearizing the diffusion equation satisfied by the macroscopic density profiles. This notion is made precise in a result called the Boltzmann-Gibbs principle.

Equation (1.6) contracts to the colour-blind density fluctuation field which we may write in the form

$$dY_t = \frac{1}{2}\triangle D(\rho_t)Y_t dt + \nabla \sqrt{[\chi D](\rho_t)}dW_t. \tag{1.8}$$

We next consider the deviations of the colour fields away from Y_t. To this end, define $U_t^i = Y_t^i - \frac{\rho_t^i}{\rho_t}Y_t$. The generalized SDE satisfied by $\boldsymbol{U}_t$ may now be written as

$$d\boldsymbol{U}_t = \frac{1}{2}\triangle S(\rho_t)\boldsymbol{U}_t dt + \nabla \sqrt{S(\rho_t)}d\boldsymbol{Z}_t, \tag{1.9}$$

where $\boldsymbol{Z}_t$ is a Gaussian process *independent* of W_t, such that $1 \cdot \boldsymbol{Z}_t = 0$. In terms of the original formulation we may express dZ_t^i as $\sqrt{\rho_t^i}dW_t^i - \frac{\rho_t^i}{\rho_t}\sum_{i=1}^k \sqrt{\rho_t^i}dW_t^i$, which is independent of $\sum_{i=1}^k \sqrt{\rho_t^i}dW_t^i = \sqrt{\rho_t}dW_t$ by direct calculation of the covariances. Thus we have that $\boldsymbol{Z}_t$ has covariance structure given by

$$E[\langle Z_t^i, f \rangle \langle Z_s^j, g \rangle] = \int_0^{\min(s,t)} \int f(x)g(x)\rho_u^i(x)\left\{\delta_{i,j} - \frac{\rho_u^j(x)}{\rho_u(x)}\right\} dx du. \tag{1.10}$$

Given (1.9) and (1.8), one may recover (1.6).

It is our belief that for general interacting particle systems, such as symmetric simple exclusion, the colour density fluctuations will be described by equations (1.8) and (1.9). It is only the special relation $[\chi D]\rho = S(\rho)\rho$ which allows the formula (1.6) to emerge for the zero range process

Using equations (1.6) and (1.8), we may also write the colour density fields as

$$dY_t^i = \frac{1}{2}\triangle S(\rho_t)Y_t^i dt + \frac{1}{2}\triangle \frac{[D-S](\rho_t)}{\rho_t}\rho_t^i Y_t dt + \nabla \sqrt{\frac{[\chi \cdot D](\rho_t)}{\rho_t}\rho_t^i}dW_t^i,$$

for $i = 1 \ldots k$. The advantage of this formula is that we can see explicitly how the density fluctuation field for any colour interacts with the environment created by the colour-blind process. If our system was formed instead by non-interacting random walks the middle term in the above formula would be equal to zero.

In the literature, results on the fluctuations of the hydrodynamic scaling limit are known as *non-equilibrium* density fluctuations. When the system is started in an equilibrium measure, the terminology equilibrium density fluctuations is used. In fact, the term "density" is often omitted in the discussion; however, we introduce it

to emphasize the difference between these central limit theorems and fluctuations for the tagged particles.

Next consider the empirical measure for the particle paths themselves. We denote the paths of the diffusively re-scaled particles as $X_i^N(\cdot)$, each taking values in $D([0,T], \mathbb{T})$. For each N we have n particles and we assume that

$$\lim_{N \to \infty} \frac{n}{N} = \bar{\rho} \tag{1.11}$$

exists. The n particles are initially randomly distributed on the unit circle. Define the particle path empirical measure as

$$\Pi_n = \tfrac{1}{n} \sum_{i=1}^n \delta_{X_i^N(\cdot)}. \tag{1.12}$$

Π_n is a random variable taking values in $M_1\{D([0,T], \mathbb{T})\}$, the space of probability measures on $D([0,T], \mathbb{T})$ endowed with the weak topology. That is, a realization of Π_n is a measure obtained by assigning mass $\frac{1}{n}$ to each observed particle path.

A known result is that Π_n satisfies a law of large numbers

$$\Pi_n \to \Pi, \tag{1.13}$$

where the limit Π is a non-random element in the space $M_1\{D([0,T], \mathbb{T})\}$. In fact, Π is the measure of a diffusion with generator

$$\frac{1}{2} S(\rho_t) \triangle, \tag{1.14}$$

where ρ_t is the previously discussed macroscopic density and S is the self diffusion coefficient. The law of large numbers in (1.13) implies that a randomly chosen tagged particle has the asymptotic distribution Π. It also implies that the particle becomes independent of the environment created by the system. This type of law of large numbers is often called *propagation of chaos* (see, for example, [Szn3]).Propagation of chaos implies a central limit theorem for the position of a randomly chosen tagged particle.

Propagation of chaos was proved for the symmetric simple exclusion process in [Qua, Rez]. First, the unique limit of Π_n is identified using the colour hydrodynamic scaling limit [Qua]. The fluctuation results of this paper imply that the hydrodynamic scaling limit for the zero range process satisfies equation (1.3). Tightness in the space $M_1\{D([0,T], \mathbb{T})\}$ would complete the result. This last fact was proved in [Rez] for symmetric exclusion, and the method presented there may be applied to the zero range process. Large deviations associated to the above law of large numbers were also studied for the symmetric simple exclusion process in $d \geq 3$ [QRV].

The next goal is to study the associated central limit theorems. That is, we wish to understand the limiting behaviour of

$$\Gamma_n = \sqrt{n}(\Pi_n - \Pi).$$

In the case of independent random walks it is easy to see that the limiting distribution of Γ_n is a mean zero Gaussian random field with the identity variance operator. When

there is dependence between the particles the limit remains Gaussian, but we expect an additional source of variation caused by the interaction. In Section 2 we discuss how one may obtain the above limit for the zero range process as a direct consequence of our non-equilibrium colour density fluctuation result. The type of convergence is convergence in finite dimensional distributions. We also calculate explicitly the variance for a class of test functions, and provide a formula for the inverse of the variance in the general case.

Hydrodynamic scaling limits and associated fluctuations are both subjects of much interest in the literature. Scaling limits and equilibrium fluctuations are well understood for many models (see for example [KL, ML, Ros, Lu, GKL]). However, there is still much work to be done to fully understand non-equilibrium fluctuations. To our knowledge there are currently only partial results for gradient models, and no known results in the non-gradient case. We adopt here Chang and Yau's [CY] proof for the Ginzburg-Landau model in dimension one to our zero range models. The proof relies on knowing the logarithmic Sobolev inequality for the inhomogeneous zero range process [Jan]. We apply this inequality in the proof of k-colour non-equilibrium density fluctuations. The main idea behind the argument is that we may think of the zero range process for the i-th colour as a process in a random environment, and our use of the logarithmic Sobolev inequality for the inhomogeneous setting reflects this notion. In order to make use of this result we make an additional technical assumption. For example, we could assume that the particle jump rate, $c(k)$, is linear for large k (see Section 3.3 for a complete discussion of the assumptions). We do not need this assumption elsewhere in this work, however, the full result holds only in this setting. The remainder of the work holds under the usual assumptions on the zero range process, namely

$$(LG) \qquad \sup_k |c(k+1) - c(k)| < \infty \qquad\qquad (1.15)$$

as well as a weak monotonicity condition

$$(M) \qquad \inf_k \{c(k+k_0) - c(k)\} > 0, \qquad\qquad (1.16)$$

for some integer k_0.

Propagation of chaos and the associated fluctuations have been studied previously for systems with mean-field interactions, [Szn1, Szn2, ST, AB, Tan]. However, the interactions between particles in the mean-field setting are weaker than for the zero-range process. The method presented here to obtain the result for finite dimensional distributions has not been previously applied to show a fluctuation result. It is based on the ideas developed in [Qua, Rez, QRV]. As the colour version of the symmetric simple exclusion process is of non-gradient type, at this time we cannot prove tagged particle fluctuations for this model.

The outline of this paper is as follows. We begin with a discussion of the relationship between nonequilibrium density fluctuations for the colour version of the process and the central limit theorem for the particle paths. This is Section 2, where we also

provide the formulae discussed above. In Sections 3 and 4, we define notation, give a complete summary of the assumptions made, and give some preliminary results which we will use in the proof of the main result. Since the proof of Theorem 1.1 is quite involved, we first explain the result for the colour-blind model. This is done in Section 5. In Section 6 we explain how to extend this to the colour version of the model.

2. Fluctuations of Particle Path Empirical Measures

A classical result from probability theory identifies the fluctuations of the empirical measure of independent random variables as a Gaussian field with identity covariance operator. More precisely, let $X_1, X_2, \ldots$ be a collection of independent and identically distributed random variables with values in some Polish space $\mathcal{X}$ and common distribution π. We define the empirical measure, $\hat{\pi}_n$, as the random measure created by putting mass $\frac{1}{n}$ at each of the n observed X_i random variables:

$$\hat{\pi}_n = \frac{1}{n} \sum_{i=1}^n \delta_{X_i}.$$

The strong law of large numbers implies that

$$\hat{\pi}_n \to \pi$$

in probability in the weak topology of probability measures on $\mathcal{X}$. To study the fluctuations of this convergence we re-scale the quantity $\hat{\pi}_n - \pi$ by $\sqrt{n}$. That is, we now consider the quantity

$$\mathbb{G}_n = \sqrt{n}(\hat{\pi}_n - \pi).$$

A consequence of the classical central limit theorem is that

$$\mathbb{G}_n \Rightarrow \mathbb{G},$$

in the sense of finite dimensional distributions, where $\mathbb{G}$ is the Gaussian random field with mean zero and identity covariance operator. That is, for each measurable function $f : \mathcal{X} \to \mathbb{R}$ in $L^2(\pi)$ such that $\int f d\pi = 0$ we have

$$< \Gamma_n, f > \Rightarrow < \Gamma, f >,$$

where $< \Gamma, f >$ is a real-valued Gaussian random variable with mean zero and variance $\int f^2 d\pi$.

The colour density fluctuations imply a similar result for the fluctuations of the empirical measure where the random variables X_i are taken to be paths of the nearest neighbour zero-range particles after diffusive re-scaling. In particular, we have a system of n interacting particles with trajectories taking random values in $D([0,T], \mathbb{T})$. One would expect that these fluctuations also converge to a Gaussian random field; however, the random variables under study are no longer independent and this non-trivial dependence structure would introduce an additional correlation in the limiting covariance.

For each N we have n particles and we assume (1.11). The n particles are initially distributed on $\mathbb{T}$ and we assume that this initial distribution satisfies both a law of large numbers and a central limit theorem. That is, we assume that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i^N(0)} = \rho_0(x)dx$$

exists weakly on $\mathbb{T}$. Notice that this implies $\int \rho_0(x)dx = 1$. We also assume that

$$\lim_{n\to\infty} \sqrt{n} \left(\frac{1}{n} \sum_{i=1}^{n} \delta_{X_i(0)} - \rho_0(x)dx \right) = \Gamma_0$$

exists, where Γ_0 is a random element of $\mathcal{H}_{-4}$. These assumptions are implied by assumptions (D1) and (F2) of Section 3.3.

Define the empirical measure as in (1.12). We next describe a heuristic approach used to identify its limit Π. Consider the time-marginals of the process Π_n. Define

$$\Pi_n(t) = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i^N(t)}.$$

From Theorem 4.6 we have that $\Pi_n(t)$ satisfies a law of large numbers and converges weakly to the solution of (1.2). We would expect that a tagged particle started out of equilibrium would also converge to a diffusion with the self diffusion coefficient $S(\rho_t)$ and a drift term caused by the evolution of the system towards equilibrium. That is, the generator of this diffusion should be

$$\mathcal{L} = \frac{1}{2} S(\rho_t) \triangle + b \cdot \nabla. \tag{2.1}$$

We next identify the drift b by noting that under the tagged limit we would still expect the time marginals of Π to evolve according to the hydrodynamic diffusion equation (1.2). That is,

$$\mathcal{L}^* \rho_t = \frac{1}{2} \nabla \cdot D(\rho_t) \nabla \rho_t.$$

Equating with the forward equation from the generator, we obtain

$$b = \frac{1}{2\rho_t} \{ \nabla S(\rho_t) \rho_t - \nabla D(\rho_t) \nabla \rho_t \}.$$

In the zero range process we have $b \equiv 0$, as can be shown through direct computation of the above quantities. That is, for the zero range process, the limiting distribution Π is equal to the measure P, where P is the law concentrated on $C([0, T], \mathbb{T})$ of a drift-free diffusion process with generator given by

$$\mathcal{L} = \frac{1}{2} S(\rho_t) \triangle.$$

We wish to study the fluctuations of the previously described law of large numbers. That is, we center and re-scale as per usual, and study the quantity

$$\Gamma_n = \sqrt{n}(\Pi_n - \Pi). \tag{2.2}$$

We have already identified Π as the measure P defined above. Theorem 1.1 implies that $<\Gamma_n, F>$ converges to a Gaussian random variable for smooth functions $F : C([0,T], \mathbb{T}) \mapsto \mathbb{R}$ of the form

$$F(z(\cdot)) = f(z(t_1), \ldots, z(t_k)),$$

where $t_1 < \ldots < t_k$. Functions of this form correspond to the question *"what are the particles doing at $\{t_1, \ldots, t_k\}$?"*. We answer the question by colouring the particles according to their behaviour at these times, and hence, we may use the fluctuation results for the colour densities.

To describe this consider first the simplest case of

$$F(z(\cdot)) = f(z_t).$$

Denote by $\mathcal{P}_{t,s}^D$ the semi-group associated to the nonpositive operator $\frac{1}{2}D(\rho_t)\triangle$. The results of Section 5 tell us that for this choice of F, $<\Gamma_n, F>$ converges to the random variable described by

$$E[e^{i<\Gamma,F>}] = \widehat{\pi}_0(\mathcal{P}_{t,0}^D f) \exp\{-\frac{1}{2}E[\left\{\int_0^t \langle \mathcal{P}_{t,s}^D f, dZ_s \rangle\right\}^2]\}, \tag{2.3}$$

where $\widehat{\pi}_0(f)$ is the characteristic function of the random variable $<\Gamma_0, f>$. In the above Z_t denotes the generalized Gaussian process with covariance

$$E_P[\langle Z_t, f \rangle \langle Z_s, g \rangle] = \int_0^{\min(s,t)} < \nabla f, [\chi D](\rho_u)\nabla g > du. \tag{2.4}$$

Indeed, this is simply a restatement of Theorem 5.8.

Consider next the case of

$$F(z(\cdot)) = \mathbb{I}[z(t_1) \in B_1]f(t), \tag{2.5}$$

where $t > t_1$ and B_1 is an open subset of $\mathbb{T}$. Following [Rez], we define

$$\begin{array}{rcl} \rho_{t_1}^1(x) & = & \rho_{t_1}(x)\mathbb{I}[x \in B_1] \\ \rho_{t_1}^2(x) & = & \rho_{t_1}(x) - \rho_{t_1}^1(x). \end{array} \tag{2.6}$$

Let $\mathcal{P}_{t,s}^S$ denote the semi-group associated to the equation $\partial_t h = \frac{1}{2}S(\rho_t)\triangle h$. We also define $\mathcal{A}_t^1$ as the operator acting on functions h as $\frac{(D-S)(\rho_t)}{\rho_t}\rho_{1,t}\triangle h$. For the zero-range process this becomes $S'(\rho)\rho^1\triangle$. Lastly, as in the above, we define Z_t^i to be two independent generalized Gaussian processes with covariance

$$E_P[\langle Z_t^i, h \rangle \langle Z_s^i, g \rangle] = \int_0^{\min(s,t)} < \nabla f, [\chi D](\rho_u)\frac{\rho_u^i}{\rho_u}\nabla g > du.$$

The results of Section 6 imply that for the choice of F given as in (2.5), $< \Gamma_n, F >$ converges weakly to the random variable described by

$$
\begin{aligned}
E[e^{i<\Gamma,F>}] \;=\; & \widehat{\pi}_0 \left(\mathcal{P}^D_{t_1,0} \mathbb{I}_{B_1} \mathcal{P}^S_{t,t_1} f + \int_{t_1}^t \mathcal{P}^D_{s,0} \mathcal{A}^1_s \mathcal{P}^S_{t,s} f \, ds \right) \qquad (2.7) \\
& \times \exp\{ -\frac{1}{2} E \left[\left\{ \int_{t_1}^t \langle \mathcal{P}^S_{t,s} f, dZ^1_s \rangle + \int_0^{t_1} \langle \mathcal{P}^D_{t_1,u} \mathbb{I}_{B_1} \mathcal{P}^S_{t,t_1} f, dZ^1_u + dZ^2_u \rangle \right. \right. \\
& \left. \left. + \int_{t_1}^t \int_0^s \langle \mathcal{P}^D_{s,u} \mathcal{A}^1_s \mathcal{P}^S_{t,s} f, dZ^1_u + dZ^2_u \rangle ds \right\}^2 \right] \}.
\end{aligned}
$$

Notice that in the above formula the distribution of $Z^1_t + Z^2_t$ is the same as the distribution of the field Z_t with covariance given in (2.4). We chose to leave the formula in this format to emphasize the relationship between the terms. This formula is obtained by formally solving the generalized Ornstein-Uhlenbeck stochastic differential equation. That we may do this is made rigorous by Theorem 5.8 together with Theorem 6.7.

We may now repeat the above to obtain convergence for general functions F for any number of k. That is,

$$
F(z(\cdot)) = \Pi_{i=1}^k \mathbb{I}[z(t_i) \in B_i] f(z(t)). \qquad (2.8)
$$

The t_i are fixed and increasing times in $[0, T]$ with $t_k < t < T$, B_k are fixed measurable subsets of $\mathbb{T}$, and f is a smooth function. Notice that each step doubles the number of colours used, and we have 2^k colours for the general case.

Except for the initial measure π_0 of Γ_0, our limits are Gaussian. If we assume that π_0 is also a Gaussian measure on $\mathcal{H}_{-4}$ then we would have that Γ is truly a Gaussian field. We do not make this assumption; however, from this point on we will say that Γ is a Gaussian random field and omit the discrepancy caused by the initial measure.

It follows from the above that we have proved weak convergence of

$$
< \Gamma_n, F > \Rightarrow < \Gamma, F >
$$

in the sense of finite dimensional distributions for a large class of functions on $C([0, T], \mathbb{T})$. We know that Γ is a mean zero Gaussian random field. We would like to identify explicitly the covariance operator of Γ for a general function G.

For a field $G : C([0, T], \mathbb{T}) \to \mathbb{R}$ define

$$
q(t, x) = E_P[G(X(\cdot))|X_t = x] \rho_t(x),
$$

where ρ_t is again the solution to (1.2). Let γ^* denote the quantity

$$
\gamma^*(t, x) = \frac{\nabla^{-1} \{ \mathcal{G}(t, \rho_t(x); q(t, x)) \}}{[\chi \cdot D](\rho_t(x))},
$$

where $\mathcal{G}$ is the functional

$$
\mathcal{G}(t, \rho, q) = \partial_t q - \frac{1}{2} \triangle [D(\rho)q].
$$

Lastly, define the operator $\mathcal{A}^*G = \int_0^T \gamma^*(t, X_t) dX_t$.

Theorem 2.1. *For any mean-zero function G in $C_b[C([0,T], \mathbb{T}], \mathbb{R}]$ into the real numbers, $< \Gamma, G >$ is a real-valued Gaussian random variable described by*

$$E[e^{i<\Gamma,G>}] = \hat{\pi}_0(F_0) \; e^{-\frac{1}{2}\mathcal{Q}_D(G)} \; e^{-\frac{1}{2}\mathcal{Q}_S(G)}. \tag{2.9}$$

$\hat{\pi}_0$ is the characteristic function of the initial field Γ_0 and $F_0 = E_P[G(X(\cdot))|X_0]$ is the projection of the function G onto the initial field. $\mathcal{Q}_D(G) = E_P[G \; \Theta_D^{-1} G]$ may be identified through the operators Θ_D on a test function F corresponding to the operator on a field G given by the quadratic form

$$E_P[(\mathcal{A}^*G)^2]. \tag{2.10}$$

Similarly, for $\mathcal{Q}_S(G) = E_P[G \; \Theta_S^{-1} G]$ may be identified as the inverse of the quadratic form

$$E_P[(G - \mathcal{A}^*G)^2]. \tag{2.11}$$

Remark 2.2. *In the case when we are working with a system of independent random walks (that is, we choose $c(k) = k$), the above formula becomes $E_P[G^2]$ as expected.*

Proof. It is enough to establish equivalence with the limiting behaviour of $< \Gamma, F >$ for functions F of the form in (2.8).

For any function G, the behaviour of $< \Gamma, G >$ may be split into three components. First, we have the behaviour of $< \Gamma, E_P[G(\cdot)|X_0] >$, which is independent and given by the initial field measure π_0. Next, we consider the evolution of the marginals

$$< \Gamma, E_P[G(\cdot)|X_t] - E_P[G(\cdot)|X_0] >,$$

whose variance is described by the operator $\mathcal{A}^*G = \mathcal{A}^*[E_P[G(\cdot)|X_t] - E_P[G(\cdot)|X_0]]$. This is the same as (1.8) and by formula (2.9) is independent of what remains. The remaining portion is the behaviour of $< \Gamma, G >$ less the time evolution of $< \Gamma, E_P[G(\cdot)|X_t] - E_P[G(\cdot)|X_0] >$. By (2.9) and (2.11) this is Gaussian with identity variance operator under the measure P. For functions of the form given in (2.8), this is simply (1.9), which is independent of (1.8). That is, consider the case of $F(z(\cdot)) = f(z_T)\mathbb{I}_B(z_s)$. Using (2.6) and the propagation of chaos results of [Rez], we have that $E_P[f(X_T)\mathbb{I}_B(X_s)|X_t = x]\rho_t(x) = E_P[f(X_T)|X_t = x]\rho_t^1(x)$, as in the latter case the indicator function is simply equal to one. It thus remains to argue that the appropriate density in this case is given by $\rho_t^i(x)\left\{1 - \frac{\rho_t^i(x)}{\rho_t(x)}\right\}$ to match (1.10). This follows from noting that $\rho_t(x) = \sum_{i=1}^k \rho_t^i(x)\mathbb{I}[\text{particle is of type } i]$, for any number of colours. Thus, we may identify Θ_D^{-1} via the variance given in (2.3), and Θ_S^{-1} is simply the identity on $G - \mathcal{A}^*G$. $\square$

Theorem 2.1 tells us that the limiting fluctuations of propagation of chaos for our model separate into the bulk density fluctuations and the *independent* fluctuations of the remaining deviations. Fluctuations of this type have been studied previously for

systems with mean-field interactions in [Szn1, Szn2, ST, AB, Tan], where the limiting behaviour is quite different. For these models, the variance operator may be written in the form $(I - \mathcal{A})^2$, with $\mathcal{A}$ described explicitly in terms of Malliavin derivatives of P. The physical source of this form for the variance operator is most easily seen in the coupling approach developed in [Szn1]. We also note that the convergence obtained in [Szn1, Szn2, ST, AB, Tan] for the fluctuation results is weak convergence in finite dimensional distributions only.

3. Notation and Assumptions.

3.1. **Notation.** We denote the set $\{0, 1/N, \ldots, (N-1)/N\}$ as $\mathbb{T}^N$, with addition defined as addition modulo one, and the continuous unit circle as $\mathbb{T}$.

Given a subset Λ of $\mathbb{T}^N$ or of $\mathbb{Z}/N\mathbb{Z}$ we write $AV_{x \in \Lambda} f(x)$ to denote the average of the function f inside the box Λ, that is,

$$AV_{x \in \Lambda} f(x) \;\; = \;\; \frac{1}{|\Lambda|} \sum_{x \in \Lambda} f(x).$$

For a metric space $\mathcal{X}$, $C^p(\mathcal{X})$ stands for the space of real-valued functions on $\mathcal{X}$ with p continuous derivatives, and $C_b(\mathcal{X})$ the space of bounded and continuous functions on $\mathcal{X}$.

For z in $\mathbb{Z}$ define $h_z : \mathbb{T} \mapsto \mathbb{R}$ by $e_z(x) = \sqrt{2} \cos(2\pi z u)$ for z positive, $e_z(x) = \sqrt{2} \sin(2\pi z u)$ for z negative, and $e_0(x) \equiv 1$. The collection $\{e_z\}$ is an orthonormal basis of $L^2(\mathbb{T})$. $< \cdot, \cdot >$ denotes the inner product of $L^2(\mathbb{T})$. The functions e_z are also eigenfunctions of the operator $I - \triangle$, with eigenvalues $\gamma_z = 1 + 4\pi z^2$. We use the notation $\mathcal{H}_m$ to denote the Hilbert space formed by taking the completion of $C^\infty(\mathbb{T})$, the space of infinitely differentiable real functions on $\mathbb{T}$, under the inner product

$$< f, g >_m = < f, (I - \triangle)^m g > .$$

For each positive integer k we denote by $\mathcal{H}_{-m}$ the dual of $\mathcal{H}_m$ relative to the inner product $< \cdot, \cdot >$.

We define the k-fold product of these space to be

$$\mathcal{H}_{-m}^k = \mathcal{H}_{-m} \times \mathcal{H}_{-m} \times \ldots \times \mathcal{H}_{-m}.$$

That is, for $\boldsymbol{f} \in \mathcal{H}_{-m}^k$ we have $\boldsymbol{f} = \{f_1, \ldots, f_k\}$ and each f_i is an element of $\mathcal{H}_{-m}$. The norm $||\boldsymbol{f}||_{-m,k}$ is defined as

$$||f_1||_{-m} + ||f_2||_{-m} + \ldots + ||f_k||_{-m}.$$

We also define the norm $||f||_*$ for a function defined on the discrete unit circle and satisfying the condition $\sum_{x \in \mathbb{T}^N} f(x) = 0$ as

$$||f||_* = \frac{1}{N} \sum_{x \in \mathbb{T}^N} f(x)[-\triangle_N^{-1} f](x),$$

where $\triangle_N$ denotes the discrete Laplacian

$$[\triangle_N h](x) = N^2\{h(x+1) - 2h(x) + h(x-1)\}.$$

We denote by $< \cdot, \cdot >_*$ the associated inner product. The traditional notation for this norm is also $|| \cdot ||_{-1}$, which we do not use to avoid confusion with the Sobolev spaces defined above. We also define the product norm to be

$$||\boldsymbol{f}||_{*,k} = ||f_1||_* + ||f_2||_* + \ldots + ||f_k||_*,$$

for $\boldsymbol{f} = \{f_1, \ldots, f_k\}$.

3.2. **The Model.** The class of zero range particles we describe is the symmetric nearest neighbour zero range interacting particle system in dimension one.

The Colour-Blind Process. First consider the evolution of the number of particles at each site. If a particle moves from site x to site y the configuration η changes to $\eta^{x,y}$ where

$$(\eta^{x,y})(z) = \begin{cases} \eta(z) - 1 & \text{if } z = x, \\ \eta(z) + 1 & \text{if } z = y, \\ \eta(z) & \text{otherwise.} \end{cases}$$

It does this at rate $\frac{1}{2}c(\eta(x))$. The system is a Markov process and we can write down its generator L as

$$(Lf)(\eta) = \frac{1}{2} \sum_{x \sim y} c(\eta(x))[f(\eta^{x,y}) - f(\eta)] \tag{3.1}$$

where $x \sim y$ denotes nearest neighbours of $\mathbb{Z}/N\mathbb{Z}$.

For $\varphi > 0$ define the partition function $Z(\varphi) = \sum_{k \geq 0} \frac{\varphi^k}{c(k)!}$. Under the assumptions (LG) and (M), the radius of convergence for $Z(\cdot)$ is infinite. Fix $0 < \varphi < \infty$ and denote by μ_φ the product measure on $\mathcal{X}$ with marginals

$$\mu_\varphi(\eta(x) = k) = \frac{\varphi^k}{Z(\varphi)\, c(k)!}$$

where $c(k)! = \prod_{1 \leq m \leq k} c(m)$ if $k > 0$ and $c(0)! = 1$. The family of measures μ_φ indexed by φ are stationary and reversible for the zero-range process. If we define the product measures on the infinite lattice $\mathbb{Z}$, and not on $\mathbb{T}$, these measures represent the full set of extremal invariant measures for the system [And].

The measures μ_φ are referred to as the grand canonical measures in the physics literature. They are not ergodic, however, as we have already pointed out that the system evolution preserves the total number of particles. If we condition on the average number of particles, y, we do obtain stationary, reversible, and ergodic measures: $\mu_\varphi(\eta|\bar{\eta} = y)$. Equivalently, we may also condition on the total number of particles. These conditional measures are called the canonical ensembles.

To emphasize that these measures are defined on the discrete unit circle of size N we will use the notation $\nu_{N,y}$. That is, $\nu_{N,y} = \mu_\varphi(\eta|\bar{\eta} = y)$. If we are considering the

measures for configurations restricted to a subset of the circle, and this subset is of size K, we will use the notation $\nu_{K,y}$. Because of the homogeneity of the system there is no ambiguity in the notation.

Let $\rho(\varphi) = E_{\mu_\varphi}[\eta(x)]$ denote the average density of particles. By straightforward computation we obtain the following identities

$$\rho(\varphi) \;=\; \varphi\, Z'(\varphi)\, Z^{-1}(\varphi), \tag{3.2}$$
$$\rho'(\varphi) \;=\; \varphi^{-1}\, E_{\mu_\varphi}[(\eta(x) - \rho(\varphi))^2] > 0. \tag{3.3}$$

As the variance must be strictly positive for $\varphi > 0$, we conclude that $\rho(\varphi)$ is a strictly increasing function and hence invertible. Because ρ has the natural interpretation of the density, we will fix ρ and think of φ as $\varphi(\rho)$. We shall then index the invariant measure by ρ: μ_ρ. Notice also that $E_{\mu_\rho}[c(\eta(x))] = \varphi(\rho)$.

The static compressibility mentioned in the introduction is in general defined as

$$\textstyle\sum_{x \in \mathbb{Z}/N\mathbb{Z}} E_{\mu_\varphi}[\eta(x); \eta(0)],$$

where we use the notation $E[f; g]$ to denote the covariance of the functions f and g.

We are now in the position where we may express the self and bulk diffusion coefficients, as well as the static compressibility explicitly.

Remark 3.1. *For the zero range process we have:*

$$S(\rho) = \frac{\varphi(\rho)}{\rho}, \quad D(\rho) = \varphi'(\rho), \quad and \quad \chi(\rho) = \frac{\varphi(\rho)}{\varphi'(\rho)}.$$

This follows from the above discussion of the invariant measure μ_ρ.

We next define the Dirichlet form, $D_{\mu_\rho}(f) = E_{\mu_\rho}[f(-L)f]$. Since μ_ρ is reversible for the dynamics, this is equivalent to

$$D_{\mu_\rho}(f) = \frac{1}{2} \sum_{x \sim y} E_{\mu_\rho}[c(\eta(x))[f(\eta^{x,y}) - f(\eta)]^2]. \tag{3.4}$$

The same identity holds when the expectation is taken with respect to the canonical ensembles, in which case we denote the Dirichlet form as $D_{\nu_{N,y}}(f)$. We may also restrict the Dirichlet form to a subset of the discrete circle, Λ_K, where $|\Lambda| = K$. Here we again use the notation $D_{\nu_{K,y}}(f)$.

$$D_{\nu_{K,y}}(f) = \frac{1}{2} \sum_{x \sim y \in \Lambda_K} E_{\nu_{K,y}}[c(\eta(x))[f(\eta^{x,y}) - f(\eta)]^2]. \tag{3.5}$$

The Colour Process. To simplify notation we define the model for the case when there are only two colours. The extension to general k is immediate.

Imagine that the zero range process is made up of two different colours of particles. The two types of particles are mechanically identical to the regular zero range process particles, but the two-colour process keeps track of the two types of particles as the system evolves. That is, we are now studying the evolution of the number of particles

of each type at site x in $\mathbb{Z}/N\mathbb{Z}$. Its elements will typically be denoted by the pair of configurations $\boldsymbol{\eta} = (\eta_1, \eta_2) \in \mathbb{N}^{\mathbb{Z}/N\mathbb{Z}} \times \mathbb{N}^{\mathbb{Z}/N\mathbb{Z}}$. The dynamics for each particle are the same as in the zero range process. Thus, the first particle of colour i to jump from site x does so at rate

$$c_i(\boldsymbol{\eta}(x)) = \frac{\eta_i(x) \; c(\eta_1(x)) + \eta_2(x))}{(\eta_1(x) + \eta_2(x))} = \eta_i(x)\frac{c(\eta(x))}{\eta(x)},$$

where $\eta(x) = \eta_1(x) + \eta_2(x)$. We write down the infinitesimal generator for the two colour process

$$(Lf)(\boldsymbol{\eta}) \;\; = \;\; \frac{1}{2}\sum_{i=1}^{2}\sum_{x \sim y} c_i(\boldsymbol{\eta}(x))[f(\boldsymbol{\eta}_i^{x,y})) - f(\boldsymbol{\eta})]. \tag{3.6}$$

Here $\boldsymbol{\eta}_i^{x,y}$ denotes the configuration obtained from $\boldsymbol{\eta}$ by moving one particle of colour i from site x to site y. Note that if the function f is "blind" to the particle colour, i.e. $f(\boldsymbol{\eta}) = f(\eta_1 + \eta_2)$, then Lf is equivalent to the generator for the previously defined zero-range process. Because of this contraction, we shall not use a different notation for the generators of the two processes.

We now define the grand canonical measures and the canonical ensembles for this process. Fix $\boldsymbol{\varphi} = \{\varphi_1, \varphi_2\} > 0$ and denote by $\mu_{\boldsymbol{\varphi}}$ the product measure on $\mathcal{X}^2$ with marginals

$$\mu_{\boldsymbol{\varphi}}(\eta_1(x) = k, \eta_2(x) = m) = \frac{\varphi_1^k \; \varphi_2^m}{c(k+m)!}\begin{pmatrix} k+m \\ k \end{pmatrix}\frac{1}{Z(\varphi_1 + \varphi_2)}.$$

$Z(\varphi)$ is the partition function defined previously. We also have that

$$\mu_{\boldsymbol{\varphi}}(\eta_1(x) = k|\eta_2(x) = m) \;\; = \;\; \frac{\varphi_1^k}{c_m(k)!Z_m(\varphi_1)}, \tag{3.7}$$

where $c_m(k) = \frac{kc(k+m)}{k+m}$ and $Z_m(\varphi_1)$ is the associated partition function. The family of measures indexed by $\boldsymbol{\varphi}$ are stationary and reversible for the two-colour zero range process.

Let $\rho^i(\boldsymbol{\varphi}) = E_{\mu_{\boldsymbol{\varphi}}}[\eta_i(x)]$ denote the density of particles of the ith colour. Notice that

$$\rho^i \;\; = \;\; \frac{\varphi_i \, Z'(\varphi)}{Z(\varphi)}, \text{ and}$$

$$\partial_{\varphi_j}\rho^i \;\; = \;\; \varphi_j^{-1}\, E_{\mu_{\boldsymbol{\varphi}}}[\eta_i(x)\eta_j(x) - \rho^i\rho^j].$$

In particular, this implies that the Jacobian of the transformation $\boldsymbol{\varphi} \mapsto \boldsymbol{\rho} = \{\rho^1, \rho^2\}$ has determinant strictly positive for $\boldsymbol{\varphi} > 0$. As before, we choose to index the invariant measure $\mu_{\boldsymbol{\varphi}}$ by the pair (ρ^1, ρ^2), (that is, work with $\mu = \mu_{\rho^1, \rho^2}$), where we now consider $\varphi_i = \varphi_i(\boldsymbol{\rho})$. The function φ_i can be recovered through $\varphi_i = E_{\mu_{\boldsymbol{\varphi}}}[c_i(\boldsymbol{\eta}(x)] = \rho^i\frac{Z'(\varphi)}{Z(\varphi)} = \frac{\rho^i}{\rho}\varphi$. Since $c(\eta_1(x) + \eta_2(x)) = c_1(\boldsymbol{\eta}(x)) + c_2(\boldsymbol{\eta}(x))$, we obtain that $\varphi = \varphi_1 + \varphi_2 = E_{\mu_{\boldsymbol{\varphi}}}[c(\eta_1(x) + \eta_2(x))]$ for the colour-blind model.

As before we define the canonical ensembles to be the measures $\mu_{\boldsymbol{\varphi}}$ conditioned on the average density of both colours of particles, which we denote as $\boldsymbol{y} = \{y_1, y_2\}$,

$$\nu_{N,\boldsymbol{y}}(\boldsymbol{\eta}) = \mu_{\boldsymbol{\varphi}}(\boldsymbol{\eta}|\overline{\eta}_i = y_i, i = 1, 2).$$

If we consider this measure only for configurations restricted to a subset of the circle, where the subset is of size K, we will use the notation $\nu_{K,\boldsymbol{y}}$.

The Dirichlet form for the two-colour version, $D_{\mu_\rho}(f) = E_{\mu_\rho}[f(-L)f]$, is then

$$\frac{1}{2}\sum_{x\sim y} E_{\mu_\rho}[c_1(\boldsymbol{\eta}(x))[f(\boldsymbol{\eta}_1^{x,y}) - f(\boldsymbol{\eta})]^2] + \frac{1}{2}\sum_{x\sim y} E_{\mu_\rho}[c_2(\boldsymbol{\eta}(x))[f(\boldsymbol{\eta}_2^{x,y}) - f(\boldsymbol{\eta})]^2]$$
$$= D_{\mu_\rho}^1(f) + D_{\mu_\rho}^2(f). \tag{3.8}$$

The same identity holds for the canonical ensembles, in which case we have the Dirichlet form

$$D_{\nu_{N,\boldsymbol{y}}}(f) = E_{\nu_{N,\boldsymbol{y}}}[f(-L)f] = D_{\nu_{N,\boldsymbol{y}}}^1(f) + D_{\nu_{N,\boldsymbol{y}}}^2(f).$$

We will use the notation $D_{\nu_{K,\boldsymbol{y}}}(f)$ as in the previous section, when we have restricted the Dirichlet form to a subset of size K.

We denote the expectation with respect to the measure induced by the zero range process started in equilibrium as E_{EQ} and as E_{NEQ} if the process is started out of equilibrium. Because of the previously noted contraction, there is no contradiction in using this notation for both the colour and colour-blind models.

3.3. **Assumptions.** We separate the assumption into several subsections.

On the Rate Function.

We make the assume that the rate function $c(\cdot)$ satisfies (LG) and (M) of (1.15) and (1.16). Assumption (LG) is necessary to ensure that the zero range process is well defined on the infinite lattice [And]. Condition (M) rules out the cases, such as the queueing system corresponding to $c(k) = \mathbb{I}(k \geq 1)$, where the spectral gap is known to depend on the density of particles.

A key ingredient in the proof of nonequilibrium density fluctuations is the logarithmic Sobolev inequality. In order to make use of this tool in the inhomogeneous zero range process we make an additional assumption. We make the assumption *only* to be able to quote this result, and it is not required in the remainder of the work.

(E). Recall the conditional grand canonical measure from (3.7). We assume that for every $N_0 \in \mathbb{N}$ there exists a finite positive constant $C = C(N_0)$ such that

$$\frac{1}{C} \leq \sqrt{r}\, \mu_{N,\rho^1=\frac{r}{|\Lambda|}}\left(\sum_{x\in\Lambda} \eta_1(x) = r \,\middle|\, \eta_2\right) \leq C, \tag{3.9}$$

holds *uniformly* over all $r \in \mathbb{N}$ and configurations η on a subset $\Lambda \subset \mathbb{Z}$ such that $|\Lambda| = N_0$.

Because of the contraction principle, it is sufficient to make this assumption for only two colours. Also, by symmetry, it follows that this assumption also holds for any colour, conditioning on the configuration of the remaining particles. This is a very technical condition, and it is satisfied if instead we make, for example, one of the following assumptions.

(E1). There exists a large constant K_0, and a positive constant θ such that for all $k \geq K_0$ the rate function c satisfies $c(k) = \theta k$.

(E2). There exists a large constant K_0, and two positive constants θ_1 and θ_2 such that for all $k \geq K_0$ the rate function c satisfies for all x

$$c(k) = \begin{cases} \theta_1\, k & \text{if } k \text{ is odd,} \\ \theta_2\, k & \text{if } k \text{ is even.} \end{cases}$$

Naturally, many other variations on these exist. For more details on this assumption see [Jan].

Remark 3.2. *The assumptions (LG) and (M) imply that there exist finite positive constants c_1 and c_2 such that for all k $c_1 k \leq c(k) \leq c_2 k$.*

On Initial Density.

Let $\boldsymbol{\zeta}_0^N$ denote a sequence of probability measures on the k-fold product of $\mathbb{N}^{\mathbb{T}_\mathbb{N}}$. For each fixed N, $\boldsymbol{\zeta}_0^N$ stands for the initial measure of the colour zero range process $\boldsymbol{\eta}_t^N$ with k colours.

(D1). We assume that for each fixed k the $\boldsymbol{\zeta}_0^N$ are associated to some fixed density profile $\boldsymbol{\rho}_0 : \mathbb{T}^k \to [0,1]$ (where $\boldsymbol{\rho}_0 = \{\rho_0^i\}_{i=1,\ldots,k}$) in the sense that

$$\lim_{N\to\infty} E_{\boldsymbol{\zeta}_0^N}\left[\left|\frac{1}{N}\sum_{x\in\mathbb{T}_\mathbb{N}} f(x/N)\eta_i^N(x) - \int_\mathbb{T} f(x)\rho_0^i(x)dx\right|\right],$$

for any continuous function $f \in C(\mathbb{T})$ and for each i from 1 to k .

(D2). We also make the assumption that

$$\limsup_{N\to\infty} E_{\boldsymbol{\zeta}_0^N}\left[AV_x\{\eta^N(x)\}^2\right] \leq C.$$

On Initial Entropy.
(H1). We assume that the relative entropy of the intitial measure grows at most linearly with respect to an invariant measure. That is, for any k, assume that there exists $\varphi* > 0$ and a constant $C = C(\mathrm{k})$ such that

$$H(\boldsymbol{\zeta}_0^N|\mu_{\boldsymbol{\varphi}^*}) \leq CN.$$

Remark 3.3. *Using the entropy inequality we can show that if the above bound is satisfied for one vector $\boldsymbol{\varphi}^*$ then it is also satisfied for each $\boldsymbol{\varphi} > 0$.*

On Initial Fluctuations.

Let $\mathbf{Y}_0^N = \{Y_{0,1}^N, \ldots, Y_{0,k}^N\}$, where each $Y_{0,i}^N$ denotes the initial density fluctuation field

$$Y_{0,i}^N(x) = \sqrt{N}\{\eta_i^N(x) - \rho_0^i(x)\}.$$

Also let $\bar{\mathbf{Y}}_0^N$ denote the centered version of $\mathbf{Y}_0^N$. That is

$$\bar{Y}_{0,i}^N = \sqrt{N}\{\eta_i^N(x) - \rho_0^i(x) - AV_x(\eta_i^N(x) - \rho_0^i(x))\}. \tag{3.10}$$

(F1). We assume the following

(a).
$$\limsup_{N\to\infty} E_{\varsigma_0^N}[N^{-1/2}||\bar{\mathbf{Y}}_0^N||_{*,k}^2] = 0$$

(b).
$$\limsup_{N\to\infty} E_{\varsigma_0^N}[N^{1/2}|\overline{\eta}_i - AV_{x\in\mathbb{T}^N}\rho_0^i|^2] = 0, \text{ for all } i = 1, \ldots, k.$$

(F2). Let $M(\mathcal{H}_{-m}^k)$ denote the space of measures on $\mathcal{H}_{-m}^k$. Assume that the initial fluctuation field has a weak limit in $M(\mathcal{H}_{-m}^k)$ for $m = 4$. That is, if we denote by P_0^N the measure induced on $M(\mathcal{H}_{-m}^k)$ by the initial field $\mathbf{Y}_t^N$, then there exists a unique measure P_0 such that $P_0^N \Rightarrow P_0$ in the weak topology of $M(\mathcal{H}_{-m}^k)$.

4. Preliminary Results

We discuss in this section some results used in the remainder of this work.

4.1. **Connection with Inhomogeneous Zero Range .** The symmetric inhomogeneous zero range process has dynamics which are the same as of the homogeneous process, except that the jump rate for a particle at site x now also depends on the site x. That is, denote the rates at site x as $c_x(\cdot)$, and $\xi(x)$ the number of particles currently at x. Then the first particle jumps from x at rate $c_x(\xi(x))$, and it jumps to one of its nearest neighbours with equal probability. We assume that these rates satisfy the bounds (LG) and (M) uniformly in the environment.

$$(LG^*) \qquad \sup_{x,k} |c_x(k+1) - c_x(k)| \leq \tilde{a}_1 < \infty$$

$$(M^*) \qquad \inf_{x,k} |c_x(k+\tilde{k}_0) - c_x(k)| \geq \tilde{a}_2 > 0$$

for some constants $\tilde{a}_1, \tilde{a}_2, \tilde{k}_0$. The generator of this process is

$$(\tilde{L}f)(\eta) = \frac{1}{2} \sum_{x\sim y\in\mathbb{Z}_N^d} c_x(\xi(x))[f(\xi^{x,y}) - f(\xi)] \tag{4.1}$$

The process has invariant grand canonical measures which are product measures with marginals

$$\tilde{\mu}_{\Lambda,\varphi}(k) = \frac{\varphi^k}{c_x(k)! \; Z_x(\varphi)}$$

where $Z_x(\varphi)$ is the appropriate partition function. The process is reversible with respect to these measures, as well as the canonical ensembles, obtained by conditioning on the average density of particles. We denote the canonical ensembles on a subset Λ as $\tilde{\nu}_{\Lambda,y}(\xi) = \tilde{\mu}_{\Lambda,\varphi}(\xi|\overline{\xi} = y)$.

Notice that the mean density varies depending on site in the inhomogeneous setting. However, we may still consider φ as an invertible function of the *overall* density $\frac{1}{|\Lambda|}\sum_{x\in\Lambda} E_{\tilde{\mu}_{\Lambda,\varphi}}[\eta(s)]$ for every fixed Λ. Similarly, we define $\sigma_x^2(\varphi) = E_{\tilde{\mu}_{\Lambda,\varphi}}[\eta(x);\eta(x)]$, and

$$\sigma_\Lambda^2(\varphi) = \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\sigma_x^2(\varphi).$$

The Dirichlet form, with expectation taken with respect to the canonical ensembles on $\Lambda_K(x) = \{y : |x - y| \leq K\}$, shall be denoted as

$$\tilde{D}_{\tilde{\nu}_{\Lambda_K(x),y}}(f) = \tfrac{1}{2}E_{\tilde{\nu}_{\Lambda_K(x),y}}\left[\sum_{x\sim y\in\Lambda_K(x)}\tilde{c}(\xi(x))(f(\xi^{x,y}) - f(\xi))^2\right].$$

We make the additional assumption that the inhomogeneous zero range satisfies the following property. For every $N_0 \geq 2$, there exists a constant $C = C(N_0)$ such that

$$(E^*) \qquad \frac{1}{C} \leq \sqrt{r}\,\mu_{\Lambda,\varphi(\frac{r}{|\Lambda|})}\left(\sum_{x\in\Lambda}\eta(x) = r\right) \leq C \qquad (4.2)$$

uniformly in $r \in \mathbb{N}$, and sets Λ for any $N_0 = |\Lambda|$.

Relationship with Colour Zero Range. Consider the colour zero range process where all except the particles of the first colour have been "frozen". By the contraction principle, we may consider only the case of two colours without loss of generality.

To this end, fix a configuration η_2 and let η_1 evolve as though it was a single-colour zero range process which uses the *non-homogenous* rate function

$$c_x(\eta_1(x)) = \frac{\eta_1(x)\; c(\eta_1(x) + \eta_2(x))}{\eta_1(x) + \eta_2(x)}.$$

This is a particular example of a non-homogeneous zero range process. The invariant measure for this process is clearly the product measure with marginals given in (3.7), the two-colour process invariant measure conditioned on the second colour configuration. It is not difficult to show that conditions (LG), (M) and (E) imply that the measures $\tilde{\mu}_x(k)$ satisfy conditions $(LG^*), (M^*)$ and (E^*).

The Dirichlet form (with respect to the canonical ensemble) in this case is simply

$$D^{x,1}_{\nu_{K,\boldsymbol{y}}(\cdot|\,\eta_2)}(f) = E_{\nu_{\Lambda_K(x),\boldsymbol{y}}(\cdot|\eta_2)}\left[\sum_{x\sim y\in\Lambda_K(x)} c_x(\eta_1(x))(f(\boldsymbol{\eta}_1^{x,y}) - f(\boldsymbol{\eta}))^2\right]. \quad (4.3)$$

We shall use the results of [Jan] on the inhomogeneous zero range process to state some useful facts about the conditional colour zero range. In what follows we shall state the results for colour 1 conditioning on colour 2, or equivalently, on all of the other colours. By symmetry, the results are valid for any colour.

4.2. Moment Bounds. The following two lemmas are proved in [LSV].

Lemma 4.1. *There exist constants C_1 and C_2, which depend only on the values a_1, a_2 and k_0, such that*

$$0 < C_1 < \frac{\sigma^2(\rho)}{\rho} < C_2 < \infty.$$

Lemma 4.2. *For all $k \geq 1$, there exists a finite constant $C(k)$ such that*

$$m_{2k}(\rho) \leq C(k)\,\sigma^{2k}(\rho)$$

for all $\rho \geq \rho_0 > 0$, where $m_j(\rho)$ denotes the jth moment of $\eta(x)$ under the distribution μ_ρ.

Next we prove a simple but useful property of the functions $\varphi(\rho)$ and $S(\rho) = \frac{\varphi(\rho)}{\rho}$:

Proposition 4.3 (Lipschitz properties). *The following functions are Lipschitz:*

  (i) $\varphi(\rho)$
  (ii) $\varphi_i(\boldsymbol{\rho})$ *as a function of ρ^i*
  (iii) $S(\rho)$

The first two functions are also strictly increasing.

Proof. The proofs follow from direct calculations of the derivatives, as well as previously computed bounds:

  (i) $\varphi'(\rho) = \frac{\varphi(\rho)}{\sigma^2(\rho)}$.
  (ii) $\partial_{\rho^1}\varphi_1(\boldsymbol{\rho}) = \frac{\rho^2}{\rho}S(\rho) + \frac{\rho^1}{\rho}\varphi'(\rho)$.
  (iii) First of all notice that if $c(k) = k$ then $S(\rho) \equiv 1$ and there is nothing to do. Otherwise, $S'(\rho)\rho = \varphi'(\rho) - S(\rho)$ which implies that $S'(\rho)$ is bounded below as long as ρ is bounded below. To finish we need to examine the limit of $S'(\rho)$ as $\rho \to 0$. This follows from noting that

$$\lim_{\rho \to 0} S'(\rho) = -\lim_{\rho \to 0} S(\rho)^2 \frac{\varphi}{\sigma^2} \times \lim_{\rho \to 0} \left\{ \frac{Z''(\varphi)Z(\varphi) - Z'(\varphi)Z'(\varphi)}{Z^2(\varphi)} \right\}.$$

$\square$

4.3. Spectral Inequalities. When the underlying measure μ is one of the canonical ensembles $\nu_{K,y}$ we will write the entropy as $H_{K,y}(f)$ for ease of notation. The following inequality was proved by Dai Pra and Posta [DPP1], [DPP2] under assumption (LG) and (M).

Theorem 4.4 (Logarithmic Sobolev Inequality). *There exists a constant $C_{LS} = C_{LS}(a_1, a_2, k_0)$ such that*

$$H_{K,y}(f) \leq C_{LS}K^2 D_{\nu_{K,y}}(\sqrt{f})$$

holds for any K, y and positive function f.

If we make the additional assumption (E), in light of the discussion of section 4.1 and the results of [Jan], we obtain a logarithmic Sobolev bound for the conditioned zero range process as well.

Theorem 4.5 (Logarithmic Sobolev Inequality for Conditioned Zero Range). *Assume that conditions (LG), (M), and (E) hold. Then there exists a constant $\tilde{C}_{LS} = \tilde{C}_{LS}(a_1, a_2, k_0)$ such that*

$$H(f|\nu_{K,\boldsymbol{y}}(\cdot|\eta_2)) \leq \tilde{C}_{LS} K^2 D_{\nu_{K,\boldsymbol{y}}(\cdot|\eta_2)}^{x,1}(\sqrt{f})$$

holds for any $\Lambda_K(x)$, y_1, density f on $\mathbb{N}^{\Lambda_K(x)}$, and configuration of particles of the second colour η_2.

4.4. **Hydrodynamic Scaling Limits.**

Theorem 4.6. *Under assumptions (LG), (M), (D1) and (D2) we have that for every $t \leq T$, for every continuous function $g : \mathbb{T}^d \to \mathbb{R}$ and for every $\delta > 0$,*

$$\lim_{N \to \infty} P_{\zeta_0^N}\left[\left| \frac{1}{N} \sum_{x \in \mathbb{Z}/N\mathbb{Z}} g(x/N)\eta_t(x) - \int_{\mathbb{T}} g(\theta)\rho_t(\theta)d\theta \right| > \delta \right] = 0$$

where $\rho_t(\theta)$ is the unique weak solution of the non-linear heat equation (1.2). Here we use $P_{\zeta_0^N}$ to denote the measure of the zero range process with initial measure ζ_0^N.

Remark 4.7. *It is possible to prove this under reduced assumptions on the rate function. It is sufficient that $c(k) \geq \theta k$ for all $k \geq 0$ and some positive constant θ.*

The proof of the above fact appears in Kipnis and Landim ([KL]) and is based on the entropy method first developed by Guo, Papanicolaou, and Varadhan [GPV].

Remark 4.8. *Theorem 1.1 implies the hydrodynamic scaling limit result for the colour version of the zero range process. However, it is possible to prove this result independently and under fewer assumptions. Most notably, assumption (E) is not necessary. We may use, for example, the nonhomogeneous spectral gap from [Jan].*

4.5. **Uniform Local Limit Theorems.** For $m \geq 0$, denote by $H_m(x)$ the Hermite polynomial of degree m:

$$H_m(x) = (-1)^m \exp\left(-\frac{x^2}{2}\right)\frac{d^m}{dx^m}\exp\left(-\frac{x^2}{2}\right).$$

Let $g_0(x)$ denote the density of a standard normal random variable, and define for $j \geq 1$

$$g_j(x) = g_0(x) \sum H_{j+2a}(x) \prod_{m=1}^{j} \frac{1}{k_m!} \left(\frac{\kappa_{m+2}}{(m+2)!\sigma^{m+2}} \right)^{k_m} \tag{4.4}$$

where the sum is taken over all nonnegative integer solutions $\{k_l\}_{l=1}^{j}$ and a such that $k_1 + 2k_2 + \ldots + jk_j = j$ and $k_1 + k_2 + \ldots + k_j = a$.

Theorem 4.9. *For all $\rho^* > 0$ and $J \in \mathbb{N}$, there exist finite constants $E_0 = E_0(\rho^*, J)$ and $A = A(\rho^*, J)$ such that*

$$\left| \sqrt{N\sigma^2(\rho)} \mu_\rho \left[\sum_{x \in \Lambda} \eta(x) = N\rho + \sigma\sqrt{N}z \right] - \sum_{j=0}^{J-2} \frac{1}{N^{j/2}} g_j(z) \right| \leq \frac{E_o}{(\sigma^2(\rho)N)^{(J-1)/2}}$$

uniformly over z and over all parameters $\rho^* \geq \rho \geq A/N$.

The above is an Edgeworth expansion for a lattice distribution, valid uniformly for the family of measures μ_ρ, the first as $A/N \leq \rho \leq \rho_0$. Its proof appears in [LSV] We also make use of the same result proved in [Jan] for the inhomogeneous process. We state the result for the conditioned zero range process. In the theorem we have that $|\Lambda| = N$.

Theorem 4.10. *For all $\rho^* > 0$ and $J \in \mathbb{N}$, there exist finite constants $E_0 = E_0(\rho^*, J)$ and $A = A(\rho^*, J)$ such that*

$$\left| \sqrt{N\sigma_\Lambda^2} \mu_{N,\rho} \left[\sum_{x \in \Lambda} \eta_1(x) = N\rho^1 + \sigma\sqrt{N}z \middle| \eta_2 \right] - \sum_{j=0}^{J-2} \frac{1}{N^{j/2}} g_j(z) \right| \leq \frac{E_0}{(\sigma_\Lambda^2 N)^{(J-1)/2}}$$

uniformly over z, configurations of η_2, and over all parameters $A/N \leq \rho^1 \leq \rho^*$.

5. Density Fluctuations in Non-equilibrium

Let ρ_t be the unique solution to the partial differential equation (1.2). Define the density fluctuation field for the colour-blind model as:

$$< Y_t^N, f > = \sqrt{N} \{ \frac{1}{N} \sum_{x \in \mathbb{T}_N} f(x)[\eta_t^N(x) - \rho_t(x)] \}. \tag{5.1}$$

Let P_N denote the measure of on $D\{[0,T], H_{-4}\}$ induced by the stochastic field Y^N with initial measure ζ_0^N.

Theorem 5.1. *Under the assumptions of Section 3.3, except for (E), P_N converges weakly in $D\{[0,T], H_{-4}\}$ to a generalized Ornstein-Uhlenbeck process characterised by the generalized stochastic differential equation*

$$dY_t = \frac{1}{2} \{ \triangle \varphi'(\rho_t) \} Y_t + \{ \nabla \sqrt{\varphi(\rho_t)} \} dW_t, \tag{5.2}$$

where W_t is the Gaussian random field $\langle W_t, f \rangle$ with covariance

$$COV[\langle W_t, f \rangle \langle W_s, g \rangle] = \min(s,t) < f, g > .$$

The process W_t is also known as a generalized Brownian motion which is "Brownian" in time and "white" in space.

Before proceeding with the proof of the above we make the quick remark that non-equilibrium density fluctuations have been proved for a specific class of zero range processes in [FPV]. The class of processes in their work is very special; the rate function they consider is $c(k) = \mathbb{I}[k \geq 1]$. In this very unique case there exists a special equivalence between this zero range process and the symmetric simple exclusion process. In this setting, the martingale equations are closed in the field Y_t^N and no Boltzmann-Gibbs argument in necessary.

5.1. **Outline of Proof.** The proof is divided into several main steps. We first identify the drift and quadratic variation of the limiting field Y_t. We then show that there is only one measure which solves this martingale problem, and that the sequence of measures P^N is tight. We first turn our attention to the drift and quadratic variation martingales. Suppose that P_{N_k} is a convergent subsequence of P_N, and for ease of notation, we denote it again by P_N. Let P denote its weak limit.

Identifying the drift and quadratic variation of Y_t which matches the equation (5.2) is equivalent to the statement that for any test function f

$$< M_t, f > = < Y_t, f > - < Y_0, f > - \int_0^t < Y_s, \tfrac{1}{2}\varphi'(\rho_s(\cdot))\triangle f > ds, \qquad (5.3)$$

is a martingale under P with quadratic variation given by

$$E_P[(< M_t, f > - < M_s, f >)^2 | \mathcal{F}_s] = \int_0^t \int \varphi(\rho_u(x))[\nabla f(x)]^2 dx du. \qquad (5.4)$$

We next compare these to the martingales under the measures P_N.

By direct calculation we know that M_t^N defined below is a martingale under P_N for any N.

$$< M_t^N, f > = < Y_t^N, f > - < Y_0^N, f > - \int_0^t < \tilde{Y}_s^N, \triangle_N f > ds \qquad (5.5)$$

where $\tilde{Y}$ denotes the fluctuation field $\sqrt{N}(c(\eta^N(x)) - \varphi(\rho(x)))$.

Comment on notation: There is a slight abuse of notation in the above as we use the same inner product notation $< f, g >$ for functions f, g defined on $\mathbb{T}$, where it is equal to $\int fg dx$, and for functions f, g defined on $\mathbb{T}^N$ where it is equal to $\frac{1}{N} \sum_{x \in \mathbb{T}^N} f(x)g(x)$. We do this only to simplify the notation.

A similar calculation gives the quadratic variation of $< M_t^N, f >$: under P_N, $N_t^N(f)$ defined below is a martingale.

$$< M_t^N, f >^2 -\frac{1}{2} \int_0^t AV_{x \in \mathbb{T}^N} \left[c(\eta_s^N(x)) + c(\eta_s^N(x + \frac{1}{N})) \right] [\nabla_N f(x)]^2 ds. \qquad (5.6)$$

It hence remains to show that the limits of M_t^N and N_t^N are consistent with (5.3) and (5.4). This is much easier for $N_t^N(f)$. The difficulty which arises in M_t^N is that expression (5.5) is not closed in the field Y^N. This will prove to be the main obstacle to overcome in the proof. We start with the fact that $< M_t^N, f >$ is a martingale, and hence we know that for all N

$$E_{P_N}[< M_t^N, f > U] = E_{P_N}[< M_s^N, f > U], \tag{5.7}$$

for all bounded, $\mathcal{F}_s$-measurable U. Here, $\mathcal{F}_t$ denotes the σ-algebra on $D\{[0,T], H_{-4}\}$ generated by $F_s(f)$ for $s \leq t$ and $f \in C^\infty(\mathbb{T})$, where $F \in D\{[0,T], H_{-4}\}$. We need to show

$$E_P[< M_t, f > U] = E_P[< M_s, f > U], \tag{5.8}$$

Comparing (5.3) with (5.5) we see that we need to replace the field $\tilde{Y}_t^N$ with $\varphi'(\rho_t)Y_t^N$. This type of result is known as the Boltzmann-Gibbs principle in the literature.

The remainder of this section will be divided in the following manner. We begin by showing that every weak limit of the measures P_N solves the martingale problem. This is subdivided into identifying the asymptotic drift and then its quadratic variation. We first consider the drift, and begin with the proof of the Boltzmann-Gibbs principle. We handle the quadratic variation in Section 5.3. In Section 5.4 we discuss the theory of Holley and Stroock with states that there is only one solution to the martingale problem. Section 5.5 is dedicated to tightness of the measures P^N. These results in combination as outlined above prove Theorem 5.1.

5.2. **Identifying the Drift Martingale.** In light of the preceding discussion in the introduction, to show that under P the drift martingale is identified through (5.3), it remains to prove the following.

Theorem 5.2 (Boltzmann-Gibbs Principle.)**.** *For functions* $f \in C^1[\mathbb{T}]$

$$\lim_{N\to\infty} E_{NEQ}\left[\left|\int_0^t < f, \sqrt{N}(c(\eta_s^N) - \varphi(\rho_s) - \varphi'(\rho_s)(\eta_s^N - \rho_s)) > ds\right|\right] \to 0. \tag{5.9}$$

This section is dedicated to the proof of this result.

The first step of the proof is to replace $\rho_t(x)$, the solution of (1.2) with the solution to the discretized version

$$\partial_t \rho = \frac{1}{2} \triangle_N \varphi(\rho), \tag{5.10}$$

with initial conditions $\rho_t(x)|_{t=0} = \rho_0(x)$ for x in $\mathbb{T}_N$. The difference between the two solutions is of order $\frac{1}{N}$ and hence does not affect (5.9), [RM]. Because of this fact and in order to simplify notation, we continue to denote the solution of (5.10) as ρ in the remainder of this section.

We begin by re-writing the field $\sqrt{N}(c(\eta_t^N) - \varphi(\rho_t) - \varphi'(\rho_t)(\eta_t^N - \rho_t))$ as the sum of five separate parts:

$$
\begin{aligned}
\Phi_1^N(t) &= \sqrt{N}[\eta_t^N(x) - \bar{c}_K(\eta_t^N(x)) - \varphi'(\rho_t(x))\{\eta_t^N(x) - m_t^K(x)\}] \\
\Phi_2^N(t) &= \sqrt{N}[\bar{c}_K(\eta_t^N(x)) - \varphi(m_t^K(x))]\mathbb{I}[m_t^K(x) \leq R] \\
\Phi_3^N(t) &= \sqrt{N}[\varphi(m_t^K(x)) - \varphi(\rho_t(x)) - \varphi'(\rho_t(x))\{m_t^K(x) - \rho_t(x)\}]\mathbb{I}[m_t^K(x) \leq R] \\
\Phi_4^N(t) &= \sqrt{N}[-\varphi(\rho_t(x)) - \varphi'(\rho_t(x))\{m_t^K(x) - \rho_t(x)\}]\mathbb{I}[m_t^K(x) > R] \\
\Phi_5^N(t) &= \sqrt{N}\bar{c}_K(x)\mathbb{I}[m_t^K(x) > R].
\end{aligned} \tag{5.11}
$$

Here, $m_t^K(x) = AV_{|y-x| \leq K/N} \eta_t^N(y)$ and similarly, $\bar{c}_K(x) = AV_{|y-x| \leq K/N} c(\eta_t^N(y))$.

To prove Theorem (5.2), we thus need to show that

$$
E_{NEQ}[|\int_0^t < f, \Phi_i^N(s) > ds|] \to 0 \tag{5.12}
$$

for for each i. We begin by proving several lemmas, which summarize the two main components of the Boltzmann-Gibbs principle. The argument is essentially a Taylor argument, and the local equilibrium principle gives us the first term in the expansion, while "equivalence of solutions" allows us to control the resulting error term.

5.2.1. *Local Equilibrium Principle.* The first lemma we need proves that over boxes of microscopic size K we are able to replace the average of a function with the expectation of said function at the average density $m^K(x)$. We take $K << N$. It turns out in the end that the correct scaling for this result is to take K to be slightly smaller than $\sqrt{N}$. To handle this we will write $K = l\sqrt{N}$, and we will let $l \to 0$.

Our main purpose in proving this result is to handle $i = 2$ in (5.12), however, we state and prove the result with slightly more generality. For a function $g : \mathbb{R} \mapsto \mathbb{R}$ define $V(x) = V_{(g,K,R)}(x)$ to be the function

$$
V(x) = (\bar{g}_K(\eta^N(x)) - \hat{g}_K(m^K(x)))\mathbb{I}[m^K(x) \leq R],
$$

where $\bar{g}_K(\eta^N(x)) = AV_{|x-y| \leq K/N} g^N(\eta(y))$, and $\hat{g}_K(\rho) = E_{\mu_{\rho = m^K(x)}}[g(\eta^N(y))]]$. If the configuration η depends on time, then V will also depend on time, and we shall write this as V_s. Notice that in the $E_{\mu_\rho}[g(\eta^N(y))]]$ we are considering configurations η on $\mathbb{T}^N$, and not on the discrete circle. However, this does not change the invariant measures, and for this reason we do not alter the notation.

Lemma 5.3 (Local Equilibrium Principle). *Suppose that g is a function satisfying $|g(x)| \leq C(1 + x)$, then*

$$
\lim_{l \to 0} \lim_{N \to \infty} \sup_{||J||_\infty < 1} N^{1/2} E_{NEQ}[|\int_0^t < J, V_s > ds|] \leq 0.
$$

Proof. Let X_s denote $<J, V_s>$. By the entropy inequality and assumption (H1), we have that the following bound for any positive β,

$$E_{NEQ}[N^{1/2}|\int_0^t X_s ds|] \leq \frac{2}{\beta N} \log E_{EQ}\left[\exp\left\{\pm\beta N \int_0^t N^{1/2} X_s ds\right\}\right] + \frac{C}{\beta}. \quad (5.13)$$

It follows from the Feynman-Kac formula that

$$E_{EQ}[\exp\{\pm\beta N \int_0^t N^{1/2} X_s ds\}] \leq \exp\{\int_0^t \Gamma_s^N ds\}, \quad (5.14)$$

where Γ_s^N is the largest eigenvalue of $\pm\beta N^{3/2} X_s + L_N$. We next reduce the span of the Dirichlet form $D_{\mu_\rho}(\sqrt{f})$. We have that

$$D_{\mu_\rho}(\sqrt{f}) \geq \frac{N}{2K+1} \inf_x \sum_{|x-z| \leq \frac{K}{N}} \sum_{|z-y| = \frac{1}{N}} E_{\mu_\rho}[c(\eta^N(x))\{\nabla_{x,y}\sqrt{f}\}^2]$$

where $\nabla_{x,y} f = f(\eta^{x,y}) - f(\eta)$. We denote the right hand side of the last line above by $\frac{N}{2K+1} \inf_x D_{\mu_\rho}^{K,x}$. We may hence bound Γ_s^N by

$$N^{3/2} \sup_f \sup_{|\alpha| \leq \beta||J||_\infty} \{E_{\mu_\rho}[\alpha \cdot V(x) \cdot f] - \frac{N^{3/2}}{2K+1} D_{\mu_\rho}^{K,x}(\sqrt{f})\}, \quad (5.15)$$

because of homogeneity of the system. We next condition on the density of particles in (5.15) and apply the logarithmic Sobolev inequality of Proposition 4.4 to obtain the bound

$$\Gamma_s^N \leq N^{3/2} \sup_f \sup_{|\alpha| \leq \beta||J||_\infty} \sup_{|y| \leq R} \{E_{\nu_{K,y}}[\alpha \cdot V(x) \cdot f] - C\frac{N^{3/2}}{K^3} H_{K,y}(f)\}, \quad (5.16)$$

for some constant C. Applying the entropy inequality again, we obtain that for any M positive the above is bounded by

$$N^{3/2} \sup_f \sup_{|\alpha| \leq \beta||J||_\infty} \sup_{|y| \leq R} \left\{ \frac{1}{M} \log E_{\nu_{K,y}}[\exp\{\alpha MV\}] + \left(\frac{1}{M} - C\frac{N^{3/2}}{K^3}\right) H_{K,y}(f) \right\} (5.17)$$

Choosing M such that $\frac{1}{M} = C\frac{N^{3/2}}{K}$ and letting $l = \frac{K}{\sqrt{N}}$, we may combine the last bound above with (5.13) to obtain

$$E_{NEQ}[N^{1/2}|\int_0^t X_s ds|] \leq \frac{CtN^{1/2}}{\beta l^3} \sup_{|\alpha| \leq \beta} \sup_{|y| \leq R} \log E_{\nu_{K,y}}[\exp\{\alpha Cl^3 V\}] + \frac{C}{\beta}. \quad (5.18)$$

The next lemma allows us to complete the argument.

Lemma 5.4 (Local Large Deviations). *For any function g which is at most linear in k, that is, $g(k) \leq C(1+k)$, and for all $y \leq R$, there exists a constant $C = C(R)$ such that for all y*

$$\log E_{\nu_{K,y}}\left[\exp\left\{\gamma AV_{|x-y| \leq K} \tilde{g}(\eta(x))\right\}\right] \leq C\left\{\frac{\gamma^2}{K} + \frac{\gamma + \gamma^2}{K^2}\right\},$$

where $\tilde{g} = g(\eta(x)) - \hat{g}(m^K(x))$.

We apply this result to $E_{\nu_{K,y}}[\exp\{\alpha Cl^3 V\}]$ and obtain that (5.18) is bounded by $C\{\beta^2 l^2 + \frac{l^{-2}+\beta^3 l^4}{N^{1/2}}\} + \frac{C}{\beta}$, from which the result follows. $\qquad\square$

Proof of Lemma 5.4. We begin by fixing a constant $M_0 > 0$. For $R \geq y \geq M_0/K$, we proceed as in [CY]. In (5.19), (5.20) and (5.21) below we look at the single site marginal of the grand canonical measures. For simplicity of notation, we omit the subscript x from the functions η. We define $\bar{g}$ to be the centered version of g, $\bar{g} = g - E_{\mu_\rho}[g]$.

First, fix θ so that $\int (\eta - \rho) e^{\theta(\eta-\rho)+\gamma/K\bar{g}(\eta)} d\mu_\rho(\eta)$ is equal to zero. A straightforward calculation shows that

$$\theta = -\frac{\gamma}{K}\frac{<\eta;g>_\rho}{<\eta;\eta>_\rho} + O(\gamma^2/K^2). \tag{5.19}$$

As we are only considering values of ρ inside a compact set, these bounds are uniform. We use the notation $<g;f>_\rho$ to denote the correlation of g and f under the measure μ_ρ. Similarly, we define $<f_1;f_2;f_3>_\rho = E_{\mu_\rho}[\bar{f}_1\bar{f}_2\bar{f}_3]$.

Next we define the measure $P(\eta|\theta,g,\rho)$ by

$$dP(\eta) = Z^{-1}(\theta,g,\rho) \exp\{\theta(\eta - \rho) + \gamma/K\bar{g}(\eta)\}d\mu_\rho(\eta),$$

with normalizing factor, $Z(\theta,g,\rho)$, given by

$$\int e^{\theta(\eta-\rho)+\gamma/K\bar{g}(\eta)}d\mu_\rho(\eta) = 1 + \gamma^2/K^2 H_2 + O(\gamma^3/K^3), \tag{5.20}$$

where $H_2 = -\frac{<\eta,g>_\rho^2}{<\eta;\eta>_\rho} + <g;g>_\rho$.

Remark 5.5. *Note that it is because of this step that we restrict ourselves to linear functions g, as otherwise the normalizing function will not be finite.*

We also define the variance under our modified measure:

$$\begin{aligned}\sigma^2(\theta,g,\rho) &= Z^{-1}(\theta,g,\rho)\int (\eta-\rho)^2 e^{\theta(\eta-\rho)+\gamma/K\bar{g}(\eta)}d\mu_\rho(\eta) \\ &= \sigma^2 Z^{-1}(\theta,g,\rho)\left\{1 - 2\gamma/KH_1 + O(\gamma^2/K^2)\right\},\end{aligned} \tag{5.21}$$

where $\sigma^2 = <\eta,\eta>_\rho$, and $H_1 = \frac{<\eta;g>_\rho<\eta;\eta;\eta>_\rho}{<\eta;\eta>_\rho^2} - \frac{<\eta;\eta;g>_\rho}{<\eta;\eta>_\rho}$. We now re-write the quantity of interest using the notation developed above.

$$\begin{aligned}E_{\nu_{K,y}}[\exp\{\gamma A V_{|x-y|\leq K}\bar{g}(\eta(x))\}] &= \frac{\int_{\bar{\eta}=\rho} e^{\theta\sum_{|x-y|\leq K}(\eta-\rho)+\gamma/K\bar{g}(\eta)}d\mu_\rho(\eta)}{\int_{\bar{\eta}=\rho} d\mu_\rho(\eta)} \\ &= Z^{-K}(\theta,g,\rho) \cdot \frac{P(\bar{\eta}=\rho|\theta,g,\rho)}{P(\bar{\eta}=\rho|\theta=g=0,\rho)}. \end{aligned} \tag{5.22}$$

We now apply Proposition 4.9 with $J = 8$ to show that

$$\frac{\sqrt{K\sigma^2(\theta, g, \rho)}P(\bar\eta = \rho|\theta, g, \rho)}{\sqrt{K\sigma^2(\rho)}P(\bar\eta = \rho|\theta = g = 0, \rho)} = 1 + O(\gamma/K^2).$$

We combine this last bound with (5.19) through (5.21) to show that (5.22) behaves like $1 + \gamma/KH_1 - 1/2\gamma^2/KH_2 + O((\gamma + \gamma^2)/K^2)$.

It remains to study the effect of considering $\tilde{g}$ vs. $\bar{g}$ in the expectation under study. Let $g^* = AV_{|x-y|\leq K}\tilde{\bar{g}}(\eta(x))$. By Jensen's inequality we have for any positive α:

$$-\alpha^{-1}\log E_{\nu_{K,y}}[\exp\{-\alpha g^*\}] \leq E_{\nu_{K,y}}[g^*] \leq \alpha^{-1}\log E_{\nu_{K,y}}[\exp\{\alpha g^*\}].$$

From previous bounds we have

$$\alpha^{-1}\log E_{\nu_{K,y}}[\exp\{\pm\alpha AV_{|x-y|\leq K}\bar{g}(\eta(x))\}] = \pm 1/KH_1 - 1/2\alpha/KH_2 + O((1+\alpha)/K^2),$$

which equals $H_1/K + O(K^2)$ if we select $\alpha = 1/K$. We combine these results to obtain the required result for the case $y \geq \frac{M_0}{K}$.

$$\gamma^{-1}\log\left(E_{\nu_{K,y}}\left[\exp\left\{\gamma AV_{|x-y|\leq K}\tilde{g}(\eta(x))\right\}\right]\right) = \gamma/KH_2 + O\left((1+\gamma)/K^2\right).$$

For $y < M_0/K$ the bound is much simpler as in this case the total number of particles, and hence any *non-local* function, is bounded. In this case the result follows by a simple Taylor expansion. $\qquad\square$

5.2.2. *Equivalence of Solutions.* By the hydrodynamic scaling limit we know that m^K and ρ solve the same limiting differential equations. We use here a standard PDE trick to show that the solutions themselves are close. This idea was first developed in [CY].

Lemma 5.6. *Under the assumptions of this chapter we have that*

$$\lim_{l\to 0}\lim_{N\to\infty} E[\int_0^t N^{-1/2}\sum_{x\in\mathbb{T}^N}(m^K(x) - \rho(x))^2 ds] = 0.$$

Proof. Let $(\triangle f)(x) = f(x+1) - 2f(x) + f(x-1)$ denote the discrete Laplacian, and write its inverse as the matrix $G = \{g(x,y)\}$. As we work with centered functions on the discrete torus, this inverse is well-defined.

A careful calculation gives that $L||\bar{Y}_t^N||_*^2$ is equal to

$$L\left\{-\frac{1}{N}\sum_{x\in\mathbb{T}^N}\sum_{y\in\mathbb{T}^N}N^{-2}g(x,y)\bar{Y}_t^N(x)\tilde{Y}_t^N(y)\right\}$$

$$= -\sqrt{N} < c(\eta^N) - \varphi(\rho), Y^N > + \sum_{x\in\mathbb{T}^N}c(\eta^N(x)) + \frac{1}{N}\sum_{x\in\mathbb{T}^N}c(\eta^N(x)). \qquad (5.23)$$

Similarly, we obtain

$$L < \bar{\eta}^N - \bar{\rho}, \bar{Y}_t^N >_* = \frac{\sqrt{N}}{2} < \eta^N - \bar{\rho}, c(\eta^N) - \varphi(\rho) > . \qquad (5.24)$$

Combining (5.23) with (5.24) we get

$$E_{NEQ}[N^{-1/2}||\bar{Y}_t^N||_*^2 + 2 < \bar{\eta}^N - \bar{\rho}, \bar{Y}_t^N >_*]$$
$$- E_{NEQ}[N^{-1/2}||\bar{Y}_0^N||_*^2 + 2 < \bar{\eta}^N - \bar{\rho}, \bar{Y}_0^N >_*]$$
$$= \quad E_{NEQ}\left[\int_0^t N^{-1/2} H_s(x)ds\right] + \quad N^{-1/2}E_{NEQ}\left[\int_0^t AV_{x\in\mathbb{T}^N}c(\eta_s^N(x))ds\right] \quad (5.25)$$

where we let $H_s(x) = c(\eta_s^N(x)) - \left\{c(\eta_s^N(x)) - \varphi(\rho_s(x))\right\}\{\eta_s^N(x) - \rho_s(x)\}$.
We next replace $H(x)$ with its local average

$$\bar{H}^K(x) = AV_{|x-y|\leq K/N}\left[c(\eta(y)) - \{c(\eta(y)) - \varphi(\rho(x))\}\{\eta(y) - \rho(x)\}\right].$$

Assumptions (LG) and (M) imply that we have finite exponential moments for the measure μ_ρ. By the entropy inequality, and assumption (H1) we conclude that

$$N^{-1/2}E_{NEQ}[\int_0^t AV_{x\in\mathbb{T}^N}c(\eta_s^N(x))ds] = O(N^{-1/2}). \qquad (5.26)$$

Thus, using summation by parts, along with $l = \frac{K}{\sqrt{N}}$, we obtain that

$$E_{NEQ}\left[\int_0^t N^{-1/2}\sum_{x\in\mathbb{T}^N}(H_s(x) - \bar{H}_s^K(x))(s)ds\right] = O(l).$$

Combining the above result with (5.25) we have

$$E_{NEQ}[N^{-1/2}||\bar{Y}_t^N||_*^2 + 2 < \bar{\eta} - \bar{\rho}, \bar{Y}_t^N >_*] - E[N^{-1/2}||\bar{Y}_0^N||_*^2 + 2 < \bar{\eta} - \bar{\rho}, \bar{Y}_0^N >_*]$$
$$= E_{NEQ}\left[\int_0^t N^{-1/2}\sum_{x\in\mathbb{T}^N}\bar{H}_s^K(x)ds\right] + O(l) + O(N^{-1/2}). \qquad (5.27)$$

We will make use of the local equilibrium principle to handle the remaining term. Define $F(\eta(y), \rho(x))$ to be the quantity

$$c(\eta(y)) - \{c(\eta(y)) - \varphi(\rho(x))\}\{\eta(y) - \rho(x)\},$$

and let $\hat{F}(m^K(x), \rho(x))$ denote its expectation under the measure $\mu_{\rho=m^K(x)}$. A brief calculation shows that $\hat{F}$ is equal to $\left\{\varphi(m^K(x)) - \varphi(\rho(x))\right\}\left\{m^K(x) - \rho(x)\right\}$. Lastly,

let $s^K(x) = AV_{|x-y| \le \frac{K}{N}} \eta^2(x)$. We split $\bar{H}^K(x)$ into three pieces:

$$
\begin{aligned}
\bar{H}^K(x) &= AV_{|x-y| \le \frac{K}{N}} \Big[F(\eta(y), \rho(x)) \\
&\qquad\qquad - \hat{F}(m^K(x), \rho(x)) \Big] \, \mathbb{I}(m^K(x) \le R \cap s^K(x) \le R_2) \\
&\quad + AV_{|x-y| \le \frac{K}{N}} \, F(\eta(y), \rho(x)) \, \mathbb{I}(m^K(x) > R \cup s^K(x) > R_2) \\
&\quad + AV_{|x-y| \le \frac{K}{N}} \, \hat{F}(m^K(x), \rho(x)) \, \mathbb{I}(m^K(x) \le R \cap s^K(x) \le R_2) \\
&= AV_{|x-y| \le \frac{K}{N}} \Big[F^{(1)}(x) + F^{(2)}(x) + F^{(3)}(x) \Big].
\end{aligned}
$$

By the local equilibrium principle the term involving $F^{(1)}$ vanishes in the limit. Even though the term $F - \hat{F}$ is not linear in η as required in the local equilibrium principle, the addition of the condition $\mathbb{I}(s^K(x) \le R_2)$ fixes the problem pointed out in Remark 5.5 . Applying bounds from Remark 3.2 of Section 3.2 we have that

$$
AV_{|y-x| \le \frac{K}{N}} F(\eta(y), \rho(x)) \le \tilde{c} \, m^K(x) - c_1 AV_{|y-x| \le \frac{K}{N}} (\eta(y) - \rho(x))^2, \tag{5.28}
$$

for some constant $\tilde{c} > 0$. If $m^K(x) > R$, we fix C and choose R so that $2||\rho||_\infty < R - C$, which implies that $m^K(x) - \rho(x) > C$ and hence (5.28) is smaller than

$$
-A \, (m(x) - \rho(x))^2 \mathbb{I}(m^K(x) > R)
$$

We next pick C large enough so that $A = c_1 - 2\tilde{c}/C > 0$.

If, on the other hand, $m^K(x) \le R$, then we control (5.28) using $s^K(x) > R_2$. For R_2 large enough

$$
AV_{|y-x| \le K/N} (\eta(y) - \rho(x))^2 > R_2 - 2R||\rho||_\infty.
$$

Hence, for $c > 0$ take $R_2 > R/c + 2R||\rho||_\infty$, we get that $R < c(R_2 - 2||\rho||_\infty R)$ implying

$$
\tilde{c} \, m^K(x) \le \tilde{c} \, R \le \tilde{c} c \, AV_{|y-x| \le K/N} (\eta(y) - \rho(x))^2.
$$

We thus obtain a bound on (5.28) of

$$
-B(m^K(x) - \rho(x))^2 \mathbb{I}(m^K(x) \le R, m_2^K(x) > R_2)
$$

where $B = c_1 - \tilde{c}c$ is positive as long as c was chosen to be sufficiently small. Thus, we obtain that for fixed R and R_2 large enough, and $\tilde{A} = min(A, B) > 0$

$$
AV_{|x-y| \le K/N} F^{(2)}(x) \le -\tilde{A}(m^K(x) - \rho(x))^2 \mathbb{I}(m^K(x) \le R, m_2^K(x) > R_2) \tag{5.29}
$$

A similar bound holds for $F^{(3)}$, using the fact that $\varphi' \ge c_1$. Combining these last bounds together with (5.27) we find

$$
E_{NEQ}[N^{-1/2}||\bar{Y}_t^N||_*^2 + 2 < \bar{\eta} - \bar{\rho}, \bar{Y}_t^N >_*] - E[N^{-1/2}||\bar{Y}_0^N||_*^2 + 2 < \bar{\eta} - \bar{\rho}, \bar{Y}_0^N >_*]
$$

$$
\le -A^* E_{NEQ}\Big[\int_0^t N^{-1/2} \sum_{x \in \mathbb{T}^N} (m^K(x) - \rho(x))^2 \mathbb{I}(m^K(x) > R \cup m_2^K(x) > R_2) ds \Big] + o(1),
$$

where $A^* = \tilde{A} + c_1$. We shall now use assumption (F1) together with the inequalities

$$\pm 2 < \bar{\eta} - \bar{\rho}, \bar{Y}_t^N >_* \leq 2N^{1/2}||\bar{\eta} - \bar{\rho}||_*^2 + \frac{1}{2}N^{-1/2}||\bar{Y}_t^N||_*^2 \tag{5.30}$$

and $||\bar{\eta} - \bar{\rho}||_*^2 \leq C||\bar{\eta} - \bar{\rho}||^2$, to conclude that

$$\frac{1}{2}N^{-1/2}E_{NEQ}[||\bar{Y}_t^N||_*^2] \leq -c\,E_{NEQ}[\int_0^t N^{-1/2}\sum_{x \in \mathbb{T}^N}(m^K(x) - \rho(x))^2 ds] + o(1).$$

The result follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

5.2.3. *Conclusion.* By Lemmas 5.3 and 5.6 we have that (5.12) holds for $i = 2, 3$. We consider Φ_1^N next. By summation by parts and (5.26) we have for $f \in C^1(\mathbb{T})$

$$|E[\int_0^t < f, \Phi_1^N(s) > ds]| \leq Cl, \tag{5.31}$$

Using arguments similar to those used to obtain (5.29) we show that for $i = 4, 5$

$$| < 1, \Phi_i^N(x) > | \leq C'(m^K(x) - \rho(x))^2 \mathbb{I}[m_t^K(x) > R] \tag{5.32}$$

for some positive constant C' and then apply Lemma 5.6. This concludes the proof of Theorem 5.2.

5.3. **Quadratic Variation.** We need to show that Y_t has quadratic variation given by (5.4). We know that under P_N, $N_t^N(f)$ defined in (5.6) is a martingale. Hence, it remains to show the following proposition. Note that it implies that test function f must be at least $C^2(\mathbb{T})$.

Proposition 5.7. *For a function h in $C^1(\mathbb{T})$.*

$$E\left[\left|\int_0^t < h, c(\eta_s^N) - \varphi(\rho_s) > ds\right|\right] \to 0. \tag{5.33}$$

Proof. This follows from the Boltzmann-Gibbs principle of Theorem 5.2, which is a stronger result, in combination with the hydrodynamic scaling limit of Proposition 4.6. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

5.4. **Unique Solution to the Martingale Problem.** We recall here a result due to Holley and Stroock [HS] which guarantees the existence of a weakly unique solution to our martingale problem.

Let $\mathcal{B}_t$ denote the nonpositive operator $\frac{1}{2}D(\rho_t)\triangle$ and $\mathcal{P}^D(t, s)$ the associated semigroup. Also, let $\mathcal{S}_t$ be the operator $\sqrt{\varphi(\rho_t)}\nabla$.

Theorem 5.8. *Fix a positive integer $m \geq 2$. Let P be a probability measure on the space $\{(C[0, T], H_{-m}), \mathcal{F}\}$, where $\mathcal{F} = \cup_{t \geq 0}\mathcal{F}_t$, and $\mathcal{F}_t$ is the canonical filtration of the process $< Y_t, f >$. Assume that for any $f \in C^\infty(\mathbb{T})$*

$$\begin{aligned} < M_t, f > &= < Y_t, f > - < Y_0, f > - \int_0^t < Y_s, \mathcal{B}_s f > ds \\ and \quad < M_t, f >^2 &- \int_0^t ||\mathcal{S}_s f||_2^2 ds \end{aligned} \tag{5.34}$$

are $L^1(P)$ martingales with respect to $\mathcal{F}_t$. Then, for all $0 \leq s \leq t$, f in $C^\infty(\mathbb{T})$, and subsets A of $\mathbb{R}$,

$$P[<Y_t, f> \in A | \mathcal{F}_s]$$

$$= \int_A \frac{1}{\sqrt{2\pi \int_s^t ||\mathcal{S}_u \mathcal{P}^D(t,u)f||_2^2 du}} \exp\left\{ -\frac{(y - <Y_s, \mathcal{P}^D(t,s)f>)^2}{2\int_s^t ||\mathcal{S}_u \mathcal{P}^D(t,u)f||_2^2 du} \right\} dy,$$

P almost surely. In particular, (5.34) together with a unique initial distribution for $<Y_0, f>$ uniquely determines P on $\{(C[0,T], H_{-m}), \mathcal{F}\}$.

Thus, to guarantee uniqueness of the limiting measure P, we need only check that we have a unique initial measure; this is simply condition (F2).

5.5. **Tightness.** Let P_N denote the probability measure on $D([0,T], \mathcal{H}_{-m})$ induced by the fluctuation field Y_t^N started in the measure ζ_0^N, and E_{NEQ} the associated expectation. In this section we prove that the sequence of probability measures P_N is tight. A consequence of the proof will be that the limit points are concentrated on the space of continuous paths $C([0,T], \mathcal{H}_{-m})$.

For a function F in $D([0,T], \mathcal{H}_{-m})$, define the (uniform) modulus of continuity, $\omega_\delta(F)$, for a fixed $\delta > 0$:

$$\omega_\delta(F) = \sup_{|s-t| \leq \delta, 0 \leq s,t \leq T} ||F_t - F_s||_{-m}.$$

To simplify notation we will often denote this supremum simply as $\sup_{s,t} ||F_t - F_s||_{-m}$. By the well-known Arzela-Ascoli result, it follows that to show the sequence P_N is tight we need to show that it satisfies two conditions:

(T1). $\lim_{M \to \infty} \limsup_{N \to \infty} P_N[\sup_{0 \leq t \leq T} ||Y_t||_{-m} > M] = 0$

(T2). $\lim_{\delta \to 0} \limsup_{N \to \infty} P_N[\omega_\delta(Y) > \epsilon] = 0 \qquad \forall \epsilon > 0.$

We will also make use of the result, due to Aldous, [Ald]. For a proof, see for example Proposition 1.6, of Section 4 in [KL]. We will state it in some generality. Let $\mathcal{X}$ denote a Polish space with metric $d(\cdot, \cdot)$. For a function g in $D([0,T], \mathcal{X})$, define the modified modulus of continuity, $\omega_\delta'(g)$, for a fixed $\delta > 0$:

$$\omega_\delta'(g) = \inf_{\{t_i\}_{i=0}^r} \max_{0 \leq i < r} \sup_{t_i \leq s < t < t_{i+1}} d(g_t, g_s),$$

where the infimum is taken over all partitions $\{t_i\}_{i=0}^r$ of $[0,T]$ such that $t_0 = 0 < t_1 < \ldots < t_r = T$ and $t_{i+1} - t_i > \delta$.

Proposition 5.9. *A sequence of probability measures P_N on $D([0,T], \mathcal{X})$ satisfies the condition*

$$\lim_{\delta \to 0} \limsup_{N \to \infty} P_N[\omega_\delta'(Y) > \epsilon] = 0$$

provided that

$$\lim_{\delta \to 0} \limsup_{N \to \infty} \sup_{\substack{\tau \in \mathcal{T} \\ \theta \leq \delta}} P_N[d(Y_\tau, Y_{\tau+\theta}) > \epsilon] = 0, \tag{5.35}$$

where $\mathcal{T}$ denotes the collection of all stopping times bounded by T.

Proposition 5.10. *The sequence of measures P_N is tight in $D([0,T], \mathcal{H}_{-m})$, for any $m \geq 4$. Moreover, all limit points are concentrated on continuous paths.*

We first consider condition (T2). Recall that $\tilde{Y}_s^N$ is the field $\sqrt{N}(c(\eta_t^N(x) - \rho_t(x))$. We may hence write

$$||Y_t^N - Y_s^N||_{-m} \leq \sup_{||f||_m \leq 1} | < \triangle_N f, \int_s^t \tilde{Y}_u^N du > | + ||M_t^N - M_s^N||_{-m}$$

where $< M_t^N, f >$ is a martingale with quadratic variation given by the martingale in (5.6). Using this result, we reduce the proof of condition (T2) to showing that both quantities

$$(T2)_A \qquad P_N \left[\sup_{s,t} \sup_{||f||_m \leq 1} | < \triangle_N f, \int_s^t \tilde{Y}_u^N du > | > \tfrac{\epsilon}{2} \right]$$

$$(T2)_B \qquad P_N \left[\sup_{s,t} ||M_t^N - M_s^N||_{-m} > \tfrac{\epsilon}{2} \right].$$

decrease as $|t - s| \leq \delta$ converges to zero. We begin with $(T2)_B$, and we split the proof into several lemmas.

Lemma 5.11 (Bounding the tails 1.). *There exists a finite constant C such that for every eigenfunction e_z of $\mathcal{H}_m$,*

$$\limsup_{N \to \infty} E_{NEQ} \left[\sup_{0 \leq t \leq T} | < M_t^N, e_z > |^2 \right] \leq CT\{1 + < \triangle e_z, \triangle e_z >\}$$

Lemma 5.12 (Bounding the tails 2.). *For $m \geq 3$,*

$$\lim_{K \to \infty} \limsup_{N \to \infty} E_{NEQ} \left[\sup_{0 \leq t \leq T} \sum_{z \geq K} \{< M_t^N, e_z >\}^2 \gamma_z^{-m} \right] = 0.$$

This last lemma implies that to prove condition $(T2)_B$ is satisfied, we need only check that for bounded K

$$\lim_{\delta \to 0} \limsup_{N \to \infty} P_N \left[\sup_{s,t} \sum_{|z| \leq K} \{< M_t^N - M_s^N, e_z >\}^2 \gamma_z^{-m} > \epsilon' \right] = 0.$$

This will follow if we can prove the following result.

Lemma 5.13. *For any f in $C^3(\mathbb{T})$*

$$\lim_{\delta \to 0} \limsup_{N \to \infty} P_N \left[\sup_{s,t} | < M_t^N - M_s^N, f > | > \epsilon' \right] = 0.$$

We make use of the following fact

$$\omega_\delta(<M_t^N, f>) \le 2\omega_\delta'(<M_t^N, f>) + \sup_t | <M_t^N, f> - <M_{t-}^N, f> |.$$

By the definition of M_t^N we know that

$$| <M_t^N, f> - <M_{t-}^N, f> | = | <Y_t^N, f> - <Y_{t-}^N, f> |,$$

and notice that because at most one particle makes a jump at any given time, this quantity is bounded above by $||f'||_\infty N^{-3/2}$. We therefore obtain that

$$\omega_\delta(<M_t^N, f>) \le 2\omega_\delta'(<M_t^N, f>) + ||f'||_\infty N^{-3/2} \tag{5.36}$$

Proof of Lemma 5.13. By (5.36) above and Proposition 5.9 it is enough to check that

$$\lim_{\delta \to 0} \limsup_{N \to \infty} \sup_{\substack{\tau \in \mathcal{T} \\ \theta \le \delta}} P_N \left[| <M_{\tau+\theta}^N - M_\tau^N, f> | > \epsilon' \right] = 0, \tag{5.37}$$

By Chebychev's inequality, and because both τ and θ are bounded stopping times, we may bound $P_N[| <M_{\tau+\theta}^N - M_\tau^N, f> | > \epsilon']$ by

$$\frac{1}{2\epsilon^2} E_{NEQ} \left[\int_0^\delta \sum_{x \in \mathbb{T}^N} [c(\eta_s^N(x))][\nabla_N f(x)^2 + \nabla_N f(x+1/N)^2] ds \right].$$

Applying the results of Section 5.3 completes the proof. $\qquad\square$

We finish the argument for $(T2)_B$ by proving Lemmas 5.11 and 5.12.

Proof of Lemma 5.11. Because M_t^N is a martingale we may use Doob's inequality to bound $E_{NEQ}[\sup_{0 \le t \le T} | <M_t^N, e_z> |^2]$ by $4E_{NEQ}[| <M_T^N, e_z> |^2]$ and this is equal to

$$\frac{4}{2} \int_0^T \sum_i \sum_{x \in \mathbb{T}^N} [c(\eta_s^N(x))][\nabla_N e_z(x)^2 + \nabla_N e_z(x-1/N)^2 + \nabla_N e_z(x+1/N)^2] ds.$$

Using the convergence established in section 5.3, and the fact that φ is bounded uniformly, we may bound the above by $T||\varphi||_\infty \{1+ <\triangle^N e_z, \triangle^N e_z>\}$. $\qquad\square$

Proof of Lemma 5.12. This follows since by Lemma (5.11) the quantity

$$E_{NEQ} \left[\sup_{0 \le t \le T} \sum_{z \ge K} \{<M_t^N, e_z>\}^2 \gamma_z^{-m} \right]$$

is bounded by $C \sum_{|z| \ge K} \gamma_z^{-m} \{1+ <\triangle^N e_z, \triangle^N e_z>\}$. $\qquad\square$

We next turn to condition $(T2)_A$:

$$(T2)_A \qquad \lim_{\delta \to 0} \limsup_{N \to \infty} P_N \left[\sup_{s,t} \sup_{||f||_m \le 1} \left| \int_s^t <\tilde{Y}_u^N, \triangle_N f> du \right| > \frac{\epsilon}{2} \right] = 0,$$

where $\tilde{Y}_s^N$ is the field $\sqrt{N}(c(\eta_t^N(x) - \varphi(\rho_t(x))))$. Similarly to the proof of the Boltzmann-Gibbs principle, we split this field up as the sum of the following terms:

$$
\begin{aligned}
\Psi_{1,t}^N &= \sqrt{N}\{c(\eta_t^N(x)) - \bar{c}_K(\eta_t^N(x))\} \\
\Psi_{2,t}^N &= \sqrt{N}\{\bar{c}_K(\eta_t^N(x)) - \varphi(m_t^K(x))\}\mathbb{I}[m_t^K(x) \le R] \\
\Psi_{3,t}^N &= \sqrt{N}\varphi'(\rho_t(x))\{m_t^K(x) - \rho_t(x)\}\mathbb{I}[m_t^K(x) \le R] \\
\Psi_{4,t}^N &= \sqrt{N}\{\varphi(m_t^K(x)) - \varphi(\rho_t(x)) - \varphi'(\rho_t(x))\{m_t^K(x) - \rho_t(x)\}\}\mathbb{I}[m_t^K(x) \le R] \\
\Psi_{5,t}^N &= -\sqrt{N}\varphi(\rho_t(x))\mathbb{I}[m_t^K(x) > R] \\
\Psi_{6,t}^N &= \sqrt{N}\bar{c}_K(x)\mathbb{I}[m_t^K(x) > R]
\end{aligned}
\tag{5.38}
$$

It hence remains to prove that for $i = 1, \dots, 6$

$$
\lim_{\delta \to 0} \limsup_{N \to \infty} P_N \left[\sup_{s,t} \sup_{||f||_m \le 1} \left| \int_s^t < \Psi_{i,u}^N, \triangle_N f > du \right| > \frac{\epsilon}{12} \right] = 0
$$

For $i \ne 2, 3$ we first apply Chebychev's inequality and use Lemma 5.9. For $i = 1$ we argue as in (5.31). For $i = 4$ we have Lemma 5.6. Lastly, we use (5.32) and Lemma 5.6 to handle $i = 5, 6$.

$$
P_N \left[\sup_{||f||_m \le 1} \left| \int_\tau^{\tau + \delta} < \Psi_{i,u}^N, \triangle_N f > du \right| > \epsilon' \right] \le \frac{1}{\epsilon'} E_{NEQ} \left[\sup_{||f||_m \le 1} \int_0^T \left| < \Psi_{i,u}^N, \triangle_N f > \right| du \right].
$$

It remains to handle the terms with $i = 2, 3$. In order to do this we use a special case of the well-known Garcia-Rodemich-Rumsey inequality.

Lemma 5.14 (Garcia-Rodemich-Rumsey inequality). *Given a function g and a strictly increasing function ψ such that $\psi(0) = 0$ and $\lim_{u \to \infty} \psi(u) = \infty$, then*

$$
|g(t) - g(s)| \le 8 \int_0^\delta \psi^{-1}\left(\frac{4B}{u^2}\right)\frac{du}{\sqrt{u}},
$$

$$
\text{where} \qquad B = \int_0^T \int_0^T \psi\left(\frac{|g(t) - g(s)|}{\sqrt{t - s}}\right) dt ds.
$$

We choose $\psi(x) = e^{N|x|} - 1$ in the above inequality. By integration by parts we obtain

$$
\int_0^\delta \log\left(1 + \frac{4B}{u^2}\right)\frac{1}{\sqrt{u}} du \le \sqrt{\delta}\{\log(1 + \frac{4B}{\delta^2}) + 4\}.
$$

To simplify notation we drop the subscripts on the field $\Psi_{i,s}^N$ and simply write it as $\Psi(s)$. We apply the Garcia-Rodemich-Rumsey inequality to any such field $\Psi(s)$ in the following way: choose $g(t) = \int_0^t < \Psi(u), f > du$, where $||f||_m \le 1$. We then obtain

$$
| \int_s^t < \Psi(u), f > du | \le \sqrt{\delta}\{\log(1 + \frac{4B}{\delta^2}) + 4\},
$$

and we may also bound B by

$$\int_0^T \int_0^T \left\{ \exp\left(N \frac{||\int_s^t \Psi(u)du||_{-m}}{\sqrt{t-s}} \right) - 1 \right\} dsdt,$$

and we have the inequality

$$A_0 = \sup_{s,t} ||\int_s^t \Psi(u)du||_{-m} \leq \frac{8}{N}\delta^{-1/2}\{\log(1 + 4B_0\delta^{-2}) + 4\}, \qquad (5.39)$$

where

$$B_0 = \int_0^T \int_0^T \exp\left(N \frac{||\int_s^t \Psi(u)du||_{-m}}{\sqrt{t-s}} \right) dsdt - T^2. \qquad (5.40)$$

It follows that

$$B_0 \leq \frac{\delta^2}{4}e^{N\epsilon/16\sqrt{\delta}} \;\Rightarrow\; A_0 \leq \epsilon/2 + \frac{64}{N},$$

and this last quantity is smaller than ϵ for large enough N. We thus have that for any field Ψ the following inequality holds

$$P_N(A_0 > \epsilon) \leq P_N\left(B_0 > \frac{\delta^2}{4}e^{N\epsilon/16\sqrt{\delta}} \right). \qquad (5.41)$$

Next, we apply the entropy inequality (see for example Prop. 8.2, of Appendix A in [KL]) to bring the problem back into equilibrium measure. We also use here assumption (H1). We thus obtain that for any set A such that $P_{EQ}^N(A) \leq e^{-NC(\delta)}$

$$\begin{aligned} P_N[A] &\leq& [\log 2 + H_N(0)][\log(1 + P_{EQ}^N(A)^{-1})]^{-1}, \\ &\leq& [CN][\log(1 + P_{EQ}^N(A)^{-1})]^{-1}, \\ &\leq& 1/C(\delta) \end{aligned} \qquad (5.42)$$

where P_{EQ}^N is the measure induced by the field Y^N started in the equilibrium measure. We have thus reduced the problem to showing

$$P_{EQ}^N\left(B_0 > \frac{\delta^2}{4}e^{N\frac{\epsilon}{16}\sqrt{\delta}} \right) \leq e^{-NC(\delta)}, \qquad (5.43)$$

where $C(\delta) \to \infty$ as $\delta \to 0$. We now apply Chebychev's inequality

$$P_{EQ}^N\left(B_0 > \frac{\delta^2}{4}\exp\left\{ N\frac{\epsilon}{16}\delta^{-1/2} \right\} \right) \leq 4\frac{E_{EQ}[B_0]}{\delta^2}\exp\left\{ -N\frac{\epsilon}{16}\delta^{-1/2} \right\}.$$

Suppose next that we have that

$$E_{EQ}[B_0] \leq e^{CN}, \qquad (5.44)$$

where C is some positive constant. Notice that $x \leq e^{-x}$ for small x, and hence if (5.44) holds, we have that there exists a $\delta_0 > 0$ and a positive constant C' depending on δ_0 such that for all $\delta < \delta_0$ we have

$$P_{EQ}^N \left(B_0 > \frac{\delta^2}{4} \exp \left\{ N \frac{\epsilon}{16} \delta^{-1/2} \right\} \right) \leq \exp \left\{ -NC' \frac{1}{\delta^2} - N \frac{\epsilon}{16} \delta^{-1/2} \right\}. \tag{5.45}$$

Hence (5.43) holds. We have in fact proved the following.

Proposition 5.15. *Suppose that there exists a constant $C > 0$ and an N_0 such that for all $N > N_0$*

$$E_{EQ}[\int_0^T \int_0^T \exp \left(N \frac{|| \int_s^t \Psi(u) du ||_{-m}}{\sqrt{t-s}} \right) ds dt] \leq e^{CN}, \tag{5.46}$$

then

$$\lim_{\delta \to 0} \limsup_N P_N \left[\sup_{s,t} || \int_s^t \Psi(u) du ||_{-m} > \epsilon \right] = 0.$$

Hence it remains to prove (5.46). We follow a similar argument presented in [CY]. By the Cauchy-Schwarz inequality

$$N || \int_s^t \Psi_u du ||_{-m} (t-s)^{-1/2} \leq \frac{a^2 N}{t-s} || \int_s^t \Psi_u du ||_{-m}^2 + \frac{N}{a^2}$$

and hence to obtain (5.46) it suffices to prove that for a constant $C = C(a) > 0$, independent of t and s,

$$E_{EQ} \left[\exp \left\{ N \frac{a^2}{(t-s)} || \int_s^t \Psi_u du ||_{-m}^2 \right\} \right] \leq e^{CN}. \tag{5.47}$$

Define Q to be a Gaussian measure defined on $\mathcal{H}_\alpha$ with zero mean and covariance $S = (1 - \triangle_N)^{-\beta}$, where $\triangle_N f = N^2 [f(x + 1/N) - 2f(x) + f(x - 1/N)]$. The existence of such a measure is guaranteed for $\beta > 1/2$ by the theory of Gaussian measures on Hilbert spaces (see for example [Var]). Define the inner product $< F, G >_k = < F, (1 - \triangle_N)^k G >$. We thus have that

$$\int \exp\{< F, G >\} dQ(G) = \int \exp\{< F, (1 - \triangle_N)^{-\alpha} G >_\alpha\} dQ(G)$$

$$= \exp \left\{ \frac{1}{2} ||F||_{-\alpha-\beta} \right\}.$$

We choose $m = \alpha + \beta$ and changing the order of integration we may write (5.47) as

$$\int E_{EQ} \left[\exp \left\{ a N^{1/2} (t-s)^{-1/2} < \int_s^t \Psi(u) du, f > \right\} \right] dQ(f) \leq e^{C(a)N}. \tag{5.48}$$

Suppose that we could show the following bound for some positive integer k

$$E_{EQ}\left[\exp\left\{aN^{1/2}(t-s)^{-1/2} < \int_s^t \Psi(u)du, f >\right\}\right] \leq e^{Ca^2\{||f||_k+1\}}. \tag{5.49}$$

Plugging condition (5.49) into (5.48) we would obtain that the left hand side of (5.48) is bounded by

$$\exp\{Ca^2\}\int \exp\{Ca^2 < f, f >_k\}dQ(f), \tag{5.50}$$

which, because Q is a Gaussian measure, integrates to

$$\{\det\left(1 - 2Ca^2(1-\triangle_N)^{k-m}\right)\}^{-1/2},$$

for a sufficiently small and $m - k > \frac{1}{2}$. Since this is (quite) smaller than e^{CN} for some positive constant C we have reduced the proof of (5.47) again to showing (5.49). That is, we have proved the following result.

Proposition 5.16. *Suppose that there exists a constant $C > 0$ and an N_0 such that for all $N > N_0$*

$$E_{EQ}[\exp\{aN^{1/2}(t-s)^{-1/2} < f, \int_s^t \Psi(u)du >\}] \leq e^{Ca^2\{||f||_k^2+1\}}. \tag{5.51}$$

then for $m - k > \frac{1}{2}$

$$E_{EQ}[\int_0^T \int_0^T \exp\left(N\frac{|| \int_s^t \Psi(u)du||_{-m}}{\sqrt{t-s}}\right)dsdt] \leq e^{CN}.$$

We thus need to show that (5.51) holds for $\Psi_{i,t}^N$ when $i = 2, 3$. To do this we will repeat the arguments of the local equilibrium principle for both terms. Again, to simplify notation, we will denote $\Psi_{i,t}^N$ simply as Ψ. Both terms satisfy the same bounds. As before, we begin with the Feynman-Kac inequality,

$$\log E_{EQ}[e^{\gamma N^{1/2} < f, \int_s^t \Psi(u)du >}] \leq \quad \delta N^{1/2}\sup_h \sup_{|y| \leq R}\left\{E_{\nu_{K,y}}[\gamma||f||_\infty \Psi(x)h] - N^{3/2}\frac{N}{K}D_{K,y}(\sqrt{f})\right\}.$$

We apply the logarithmic Sobolev inequality given in Theorem 4.4, and the entropy inequality as in (5.16) and (5.18) to bound the line above by

$$\delta N^{1/2}\sup_{|y| \leq R}\left\{\frac{N^{5/2}}{K^3}\log E_{\nu_{K,y}}\left[\exp\{||f||_\infty \gamma K^3/N^{5/2}\Psi(x)\}\right]\right\},$$

where $\gamma = \frac{a}{\sqrt{t-s}}$. We next apply Lemma 5.4 and obtain that this is smaller than $C(l, N_0, a_0)(||f||_\infty + ||f||_\infty^2)a^2$ for all $a > a_0$, and $N > N_0$. Because we are on the unit circle $\mathbb{T}$ we have, using the Sobolev inequality $\sup_{x \in \mathbb{T}} f(x) \leq \int_{\mathbb{T}} |f'(x)|dx$, for some new constant $C(l)$

$$\log E_{EQ}[\exp\{aN^{1/2}(t-s)^{-1/2} < \triangle_N f, \int_s^t \Psi(u)du >\}] \quad \leq \quad C(l)a^2\{1 + ||f||_3\}.$$

Combining this with Remark 5.15 and Remark 5.16 we may conclude that for $i = 2, 3$

$$E_{EQ}[\int_0^T \int_0^T \exp\left(N(t-s)^{-1/2}|| \int_s^t \Psi_i(u) du ||_{-m}\right) ds dt] \leq e^{CN}. \qquad (5.52)$$

This concludes the proof of condition (T2).

By Chebychev's inequality as well assumption (F2), to prove that condition (T1) is satisfied it is enough to show that

$$\lim_{M \to \infty} P_N \left[\sup_t \sup_{||f||_m \leq 1} \{< \triangle_N f, \int_0^t \tilde{Y}_u^N du >\} > M\right] = 0 \qquad (5.53)$$

$$\text{and} \quad E_{NEQ} \left[\sup_t ||M_t^N||_{-m}\right] < \infty \qquad (5.54)$$

We begin with the latter. By Lemma 5.12 it is enough to prove that

$$E_{NEQ} \left[\sup_t < M_t^N, f >\right] < \infty.$$

This follows from Doob's inequality and the result of Section 5.3.

The case (5.53) will be a repeat of the argument for $(T2)_A$. We split the field $\tilde{Y}$ as in (5.38), and show that

$$\lim_{M \to \infty} P_N \left[\sup_t \sup_{||f||_m \leq 1} \{< \triangle_N f, \int_0^t \Psi_{i,u}^N du >\} > M\right] = 0.$$

for $i = 1, \ldots, 6$. For $i \neq 2, 3$ we first apply Chebychev's inequality. For $i = 1$ we argue as in (5.31). For $i = 4$ we use Lemma 5.6. Lastly, we use (5.32) and Lemma 5.6 to handle $i = 5, 6$. For the two remaining cases, it turns out that we have already done all of the work. We again use the Garcia-Rodemich-Rumsey inequality, Proposition 5.14. We need simply to change the roles of δ and ϵ. Again, we first argue for a general field Ψ. Let A_0 and B_0 be as in (5.39) and (5.40). From (5.41) and (5.42) it follows that for any field Ψ, if we can show that

$$P_{EQ}^N(B_0 > \frac{T^2}{4} e^{N\epsilon/16\sqrt{T}}) \leq e^{-NC(\epsilon)},$$

where $C(\epsilon) \to \infty$ as $\epsilon \to \infty$, then

$$\lim_{\epsilon \to \infty} \limsup_N P_N[\sup_t || \int_0^t \Psi(u) du ||_{-m} > \epsilon] = 0, \qquad (5.55)$$

By Chebychev's inequality, it remains to show that $E_{EQ}[B_0] \leq e^{CN}$ for some positive and finite constant C. However, this is exactly the content of (5.52). This concludes the proof of tightness. Notice again that we have in fact proved the following.

Proposition 5.17. *Suppose that $E_{EQ}[B_0] \leq e^{CN}$, with B_0 defined in (5.40), for some positive and finite constant C, then*

$$\lim_{\epsilon \to \infty} \limsup_N P_N \left[\sup_t || \int_0^t \Psi(u) du ||_{-m} > \epsilon\right] = 0.$$

6. Colour Density Fluctuations

In this section we prove Theorem 1.1. For the sake of brevity, we shall write out the proof for the case when $k = 2$; the generalization to any value of k is immediate. As for the colour-blind case, a key ingredient in the proof is the logarithmic Sobolev inequality. We use the additional condition (E) in order to have the logarithmic Sobolev inequality, and the remainder of the work is valid without this assumption.

The general outline of the proof is the same as the approach of the previous section. We show that the sequence P_N is tight. We then analyze the martingales under possible weak limits of P_N and use them to identify the form of the martingales under the limiting measure. We finish by showing that there is only one measure which solves the martingale problem.

It is simpler to work with the following version of (1.6), for any smooth test functions f_i

$$< dY_{i,t}, f_i > = \quad < Y_{i,t}, \partial_{\rho^i}\varphi_i f'' > dt + < Y_{j,t}, \partial_{\rho^j}\varphi_i f'' > dt + < dW_t^i, \sqrt{\varphi_i} f_i' > \quad (6.1)$$

We begin by identifying the martingales. For $\boldsymbol{f}$ in H_4^2 define $< \boldsymbol{Y}_t^N, \boldsymbol{f} >$ to the pair $\{< Y_{t,1}^N, f_1 >, < Y_{t,2}^N, f_2 >\}$. Under P_N we have that

$$< \boldsymbol{M}_t^N, \boldsymbol{f} > = < \boldsymbol{Y}_t^N, \boldsymbol{f} > - < \boldsymbol{Y}_0^N, \boldsymbol{f} > - \int_0^t < \tilde{\boldsymbol{Y}}_s^N, \triangle \boldsymbol{f} > ds \quad (6.2)$$

is a martingale, where $\tilde{\boldsymbol{Y}}$ denotes the coupled fluctuation field created by the pair of fields $\sqrt{N} \left(c_i(\boldsymbol{\eta}(\cdot)) - \varphi_i(\boldsymbol{\rho}(\cdot)) \right)$.

The outline of this section is as follows. In Section 6.1 we identify the drift martingale under the limit of any converging subsequence of the measures P_N. To do this we prove the Boltzmann-Gibbs principle. We next handle the quadratic variation and show that there is a unique solution to the martingale problem. We finish the section by proving that the sequence of measures P_N is tight. As in the previous section this completes the proof of Theorem 1.1.

6.1. **Identifying the Drift.** Arguing as in the previous section in (5.7) and (5.8), to show that the limiting martingales (6.2) are the same as the drift martingale of (6.1), it is enough to prove the following.

Theorem 6.1 (Boltzmann-Gibbs). *For any f in $C^1(\mathbb{T})$ and $i = 1, 2$*

$$\limsup_{N\to\infty} E_{NEQ}\left[\left| \int_0^t < f, \sqrt{N}(c_i(\boldsymbol{\eta}_s^N) - \varphi_i(\boldsymbol{\rho}_s) - \nabla\varphi_i(\boldsymbol{\rho}_s) \cdot (\boldsymbol{\eta}_s^N - \boldsymbol{\rho}_s)) > ds \right| \right] \to 0.$$

The first step of the proof is to replace $\boldsymbol{\rho}_t(x)$, the weak solution of $\partial_t\boldsymbol{\rho} = \frac{1}{2}\nabla\boldsymbol{D}_2(\boldsymbol{\rho})\nabla\boldsymbol{\rho}$ with the solution to the discretized version

$$\partial_t\boldsymbol{\rho} = \frac{1}{2}\nabla_N\boldsymbol{D}_2(\boldsymbol{\rho})\nabla_N\boldsymbol{\rho} \quad (6.3)$$

with initial conditions $\boldsymbol{\rho}_t(x)|_{t=0} = \boldsymbol{\rho}_0(x)$ for x in $\mathbb{T}^N \times \mathbb{T}^N$. The difference between the two solutions is of order $\frac{1}{N}$ [RM] and hence does not affect our calculations. Because

of this fact and in order to simplify notation, we continue to denote the solution of (6.3) as $\boldsymbol{\rho}$ in the remainder of this section.

We proceed with a proof of the two main ingredients: the local equilibrium principle and "equivalence of solutions". Notice the similarity to the proofs of the previous section. This happens because the jump rates as well as their expectations for the individual types are of the form $\varphi_i = \rho^i \frac{\varphi(\rho)}{\rho}$, exhibiting linear behaviour only in the density ρ^i.

6.1.1. *Local Equilibrium Principle.* For a function $g : \mathbb{N}_+ \times \mathbb{N}_+ \mapsto \mathbb{R}$ define

$$
\begin{aligned}
V(x) = V_{(g,K,R)}(x) & = AV_{|y-x| \leq K/N} \left\{ g(\boldsymbol{\eta}^N(y)) - E_{\mu_{\rho=\boldsymbol{m}^K(x)}}[g(\boldsymbol{\eta}^N(y))] \right\} \mathbb{I}[m^K(x) \leq R] \\
& = \left\{ \bar{g}_K(\boldsymbol{\eta}^N(x)) - \hat{g}_K(\boldsymbol{m}^K(x)) \right\} \mathbb{I}[m^K(x) \leq R],
\end{aligned}
$$

with $m^K(x)$ defined as in the previous section.

Lemma 6.2. *Suppose that g is a function satisfying $|g(k_1, k_2)| \leq C(1 + k_i)$, for all (k_1, k_2) and for some i, then*

$$
\lim_{l \to 0} \lim_{N \to \infty} \sup_{||J||_\infty < 1} N^{1/2} E_{NEQ}[|\int_0^t < J, V_s > ds|] \leq 0.
$$

Without loss of generality, we will consider only the case $g(k_1, k_2) \leq C(1 + k_1)$ in the proof. We begin with a lemma. As in the previous section we state the next result for configurations η prior to the diffusive re-scaling. For g as in Lemma 6.2, define $\tilde{g} = g - E_{\mu_{\rho^i = m_i^K(x), i=1,2}(\cdot|\eta_2)}[g]$, where $m_i^K(x) = AV_{|x-y| \leq K} \eta_i(x)$.

Lemma 6.3 (Inhomogeneous Local Large Deviations)**.** *There exists a positive constant C independent of the configuration η_2 such that*

$$
\log E_{\nu_{K,\boldsymbol{y}}(\cdot|\eta_2)} \left[\exp \left\{ \gamma AV_{|x-y| \leq K} \tilde{g}(\boldsymbol{\eta}(x)) \right\} \right] \leq C \left\{ \frac{\gamma^2}{K} + \frac{\gamma + \gamma^2}{K^2} \right\}.
$$

Proof. We first note that η_1 conditioned on η_2 satisfies the conditions of a nonhomogeneous zero-range model as was shown in Section 4.1, and hence we have the necessary local limit result from Theorem 4.10 with bounds which are independent of η_2 (and hence the site x). We then repeat the same argument as in the proof of Lemma 5.4. $\qquad\square$

Proof of Lemma 6.2. Let $X(s)$ denote $< J, V_s >$, as before. Most of the proof is the same here as in the colour-blind case, and hence we omit many of the details. By the entropy inequality and Feynman-Kac formula, we have

$$
E_{NEQ}[N^{1/2}|\int_0^t X(s)ds|] \leq \frac{2}{\beta N} t \sup_s \Gamma^N(s) + \frac{C}{\beta}, \tag{6.4}
$$

where $\Gamma^N(s)$ is the largest eigenvalue of $\pm\beta N^{3/2}X(s) + L_N$. The biggest difference in this setting is that we need to introduce an additional conditioning. We reduce the span of the Dirichlet form as in (5.15) and condition on the average density *and* the particles of colour two to obtain that $\Gamma^N(s)$ is bounded above by

$$N^{3/2}\sup_f \sup_{|\alpha|\leq\beta||J||_\infty} \sup_{|\boldsymbol{y}|\leq\boldsymbol{R}} \sup_{\eta_2}\{E_{\tilde{\nu}_{\Lambda_K(x),y_1}}[\alpha\cdot V\cdot f] - \frac{N^{3/2}}{2K+1}D^{K,x,1}_{\nu_{K,\boldsymbol{y}}(\cdot|\eta_2)}(\sqrt{f})\}$$

where $D^{x,1}_{\nu_{K,y_1,y_2}}(f)$ defined in (4.3). Using the logarithmic Sobolev inequality (Theorem 4.5), the entropy inequality and Lemma (6.3) as in the colour-blind proofs we conclude that

$$E_{NEQ}[N^{1/2}|\int_0^t X(s)ds|] \leq C\{\beta^2 l^2 + \frac{l^{-2} + \beta^3 l^4}{N^{1/2}}\} + \frac{C}{\beta},$$

from which the result follows.   □

6.1.2. *Equivalence of Solutions.* Define $\bar{Y}^N_{i,t}$ to denote the centered version of the function $Y^N_{i,t}$ as in (3.10). Because of the symmetry of the process, we only consider $||\bar{Y}^N_{1,t}||^2_*$. The same formula holds as in the colour-blind case:

$$E_{NEQ}[N^{-1/2}||\bar{Y}^N_{1,t}||^2_* + 2 < \bar{\eta}^N_1 - \bar{\rho}_1, \bar{Y}^N_{1,t} >_*]$$
$$- E_{NEQ}[N^{-1/2}||\bar{Y}^N_{1,0}||^2_* + 2 < \bar{\eta}^N_1 - \bar{\rho}_1, \bar{Y}^N_{1,0} >_*]$$
$$= E_{NEQ}[\int_0^t \{- < c_1(\boldsymbol{\eta}^N) - \varphi_1(\boldsymbol{\rho}), \bar{Y}^N_{1,s} > + N^{-1/2}(1 + 1/N) \sum_{x\in\mathbb{T}^N} c_1(\boldsymbol{\eta}^N_s(x))\}ds]$$
$$= E_{NEQ}[\int_0^t N^{1/2}AV_{x\in\mathbb{T}^N}H_{1,s}(x)ds] + N^{-1/2}E[\int_0^t AV_{x\in\mathbb{T}^N}c_1(\boldsymbol{\eta}^N_s(x))ds], \qquad (6.5)$$

where $H_1(x) = c_1(\boldsymbol{\eta}^N(x)) - (c_1(\boldsymbol{\eta}^N(x)) - \varphi_1(\boldsymbol{\rho}(x)))(\eta^N_1(x) - \rho^1(x))$.

We next replace $H_1(x)$ with

$$\bar{H}^K_1(x) = AV_{|x-y|\leq K/N}[c_1(\boldsymbol{\eta}^N(y)) - \{c_1(\boldsymbol{\eta}^N(y)) - \varphi_1(\boldsymbol{\rho}(x))\}\{\eta^N_1(y) - \rho^1(x)\}].$$

By integration by parts and (5.26) we conclude that the error produced by this replacement is of the order of $O(l)$. Line (6.5) is thus equal to

$$E_{NEQ}[\int_0^t N^{1/2}AV_{x\in\mathbb{T}^N}\bar{H}^K_{1,s}(x)ds] + O(l) + O(N^{-1/2}). \qquad (6.6)$$

Next define $F_1(\boldsymbol{\eta}^N(y), \boldsymbol{\rho}(x)) = c_1(\boldsymbol{\eta}^N(y)) - \{c_1(\boldsymbol{\eta}^N(y)) - \varphi_1(\boldsymbol{\rho}(x))\}\{\eta^N_1(y) - \rho^1(x)\}$, and split $\bar{H}^K(x)$ into three pieces:

$$\begin{aligned}
\bar{H}^K_i &= AV_{|x-y|\leq K/N}[F_1(\boldsymbol{\eta}^N(y), \boldsymbol{\rho}(x)) - \hat{F}_1(\boldsymbol{m}^K(x), \boldsymbol{\rho}(x))]\mathbb{I}(m^K \leq R)\mathbb{I}(s^K_1(x) \leq R_1) \\
&\quad + AV_{|x-y|\leq K/N}[F_1(\boldsymbol{\eta}^N(y), \boldsymbol{\rho}(x))]\mathbb{I}(m^K > R \cup s^K_1(x) > R_1) \\
&\quad + AV_{|x-y|\leq K/N}[\hat{F}_1(\boldsymbol{m}^K(x), \boldsymbol{\rho}(x))]\mathbb{I}(m^K \leq R)\mathbb{I}(s^K_1(x) \leq R_1) \\
&= AV_{|x-y|\leq K/N}[F^{(1)}_1(x) + F^{(2)}_1(x) + F^{(3)}_1(x)],
\end{aligned}$$

where

$$\hat{F}_1(\boldsymbol{m}^K(x), \boldsymbol{\rho}(x)) = E_{\mu_{\boldsymbol{\rho}=\boldsymbol{m}^K(x)}}[F_1(\boldsymbol{\eta}^N(y), \boldsymbol{\rho}(x))]$$
$$= -(\varphi_1(\boldsymbol{m}^K(x)) - \varphi_1(\boldsymbol{\rho}(x)))(m_1^K(x) - \rho^1(x))],$$

and $s_1^K(x) = AV_{|x-y| \le K/N} \{\eta_1^N(x)\}^2$.

By the local equilibrium principle the first term vanishes in the limit. Even though the term F is not linear in η as required in the local equilibrium principle, the addition of the condition $\mathbb{I}(s_1^K(x) \le R_1)$ fixes the problem originally pointed out in Remark 5.5. By an identical argument to the one presented in the previous section, for fixed R and R_1 large enough there exists a constant $\tilde{A}_2 > 0$ such that the following bound holds:

$$AV_{|x-y| \le K/N} F_1^{(2)}(x) \le -\tilde{A}_2(m_1^K(x) - \rho^1(x))^2 \mathbb{I}(m^K(x) > R \cup s_1^K(x) > R_1). \quad (6.7)$$

We handle $F_1^{(3)}$ similarly. In this case we make use of the fact that the function $\varphi_1(\rho)$ is strictly increasing (Proposition 4.3). We thus have that there exists an $\tilde{A}_3 > 0$ such that:

$$AV_{|x-y| \le K/N} F^{(3)}(x) \le -\tilde{A}_3(m_1^K(x) - \rho^1(x))^2 \mathbb{I}(m^K(x) \le R) \mathbb{I}(s_1^K(x) \le R_1). \quad (6.8)$$

Combining the arguments from (6.6) through (6.8), we find that

$$E_{NEQ}[N^{-1/2} ||\bar{Y}_{t,1}^N||_*^2 + 2 < \bar{\eta}_1 - \bar{\rho}_1, \bar{Y}_{t,1}^N >_*] - E_{NEQ}[N^{-1/2} ||\bar{Y}_{0,1}^N||_*^2 + 2 < \bar{\eta}_1 - \bar{\rho}_1, \bar{Y}_{0,1}^N >_*]$$

$$\le -\tilde{A}_2 \wedge \tilde{A}_3 E_{NEQ}[\int_0^t N^{-1/2} \sum_{x \in \mathbb{T}^N} (m_1^K(x) - \rho^1(x))^2 ds] + O(l^{-1}) + o(1) \quad (6.9)$$

Using the bound (5.30) together with assumptions (F1) on (6.9) as in the colour-blind proof, we conclude

Theorem 6.4.

$$\lim_{l \to 0} \lim_{N \to \infty} E_{NEQ}[\int_0^t N^{-1/2} \sum_{x \in \mathbb{T}^N} (m_i^K(x) - \rho^i(x))^2 ds] = 0, \quad i = 1, 2.$$

We are now ready to prove Theorem 6.1. Define the field $\boldsymbol{\Phi}^N(t) = \{\Phi_1^N(t), \Phi_2^N(t)\}$, with $\Phi_1^N(t)$ given by

$$\sqrt{N} \left\{ c_1(\boldsymbol{\eta}_t^N) - \varphi_1(\boldsymbol{\rho}_t) - \partial_1 \varphi_1(\boldsymbol{\rho}_t)(\eta_{1,t}^N - \rho_{1,t}) - \partial_2 \varphi_1(\boldsymbol{\rho}_t)(\eta_{2,t}^N - \rho_{2,t}) \right\},$$

and $\Phi_2^N(t)$ defined similarly.

As before, we split $\boldsymbol{\Phi}^N$ into five separate parts.

$$\boldsymbol{\Phi}^N(t) = \boldsymbol{\Phi}_1^N(t) + \boldsymbol{\Phi}_2^N(t) + \boldsymbol{\Phi}_3^N(t) + \boldsymbol{\Phi}_4^N(t) + \boldsymbol{\Phi}_5^N(t),$$

where the first entry in each is given by

$$
\begin{aligned}
N^{-1/2}\Phi_{1,1}^N &= c_1(\boldsymbol{\eta}^N(x)) - \bar{c}_1^K(\boldsymbol{\eta}^N(x)) \\
&\quad -\partial_1\varphi_1(\boldsymbol{\rho}(x))\{\eta_1^N(x) - m_1^K(x)\} - \partial_2\varphi_1(\boldsymbol{\rho}(x))\{\eta_2^N(x) - m_2^K(x)\} \\
N^{-1/2}\Phi_{1,2}^N &= \{\bar{c}_1^K(\boldsymbol{\eta}^N(x)) - \varphi_1(\boldsymbol{m}^K(x))\}\mathbb{I}[m^K(x) \le R] \\
N^{-1/2}\Phi_{1,3}^N &= \big[\varphi_1(\boldsymbol{m}^K(x)) - \varphi_1(\boldsymbol{\rho}(x)) \\
&\quad -\partial_1\varphi_1(\boldsymbol{\rho}(x))\{m_1^K(x) - \rho^1(x)\} \\
&\quad -\partial_2\varphi_1(\boldsymbol{\rho}(x))\{m_2^K(x) - \rho^2(x)\}\big]\,\mathbb{I}[m^K(x) \le R] \\
N^{-1/2}\Phi_{1,4}^N &= \big[-\varphi_1(\boldsymbol{\rho}(x)) \\
&\quad -\partial_1\varphi_1(\boldsymbol{\rho}(x))\{m_1^K(x) - \rho^1(x))\} \\
&\quad -\partial_2\varphi_1(\boldsymbol{\rho}(x))\{m_2^K(x) - \rho^2(x)\}\big]\,\mathbb{I}[m^K(x) > R] \\
N^{-1/2}\Phi_{1,5}^N &= \bar{c}_1^K(x)\mathbb{I}[m^K(x) > R],
\end{aligned}
$$

with $\bar{c}_1^K(x) = AV_{|x-y|\le\frac{K}{N}}c_1(\eta(y))$. By symmetry, we only work with the $\Phi_1^N(t)$. By the local equilibrium principle and Theorem 6.4 we already have for $i = 2$ and 3 that

$$
E_{NEQ}[|\int_0^T <\Phi_{1,i}^N, f> ds|] \to 0, \tag{6.10}
$$

for any continuous function f. The result for $i = 5$ follows from the same argument for the colour blind case, because $\bar{c}_1^K(x) \le \bar{c}^K(x)$. Next, we consider $i = 4$. We may pick R sufficiently large so that

$$
|\Phi_4^N(t)| \le CN^{-1/2} \sum_{i=1,2}^N (m_i^K(x) - \rho^i(x))^2
$$

and hence (6.10) holds also for $i = 4$. It remains to consider $i = 1$. An argument using summation by parts like the one presented in the previous section proves the result. Notice that it is in this part of the argument that we require the most smoothness of the function f. It is enough to consider test functions f which have a bounded first derivative. This concludes the proof of the Boltzmann-Gibbs principle.

6.2. **Quadratic Variation.** We begin with a lemma. Let $K << N$, fix R large, and define

$$
X_{t,i}^K(x) = AV_{|x-y|\le\frac{K}{N}}\{c_i(\boldsymbol{\eta}_t^N(x)) - \varphi_i(\boldsymbol{\rho}_t(x))\}\mathbb{I}[m^K(x) \le R].
$$

Lemma 6.5. *Assume that $|t - s| < T$. For any choice of $l = \frac{K}{\sqrt{N}}$ there exists a constant $C = C(l, R)$ so that the following bound holds uniformly for $||J||_\infty \le 1$:*

$$
\log E_{EQ}[\exp\{\gamma N < J, \int_s^t X_{u,i}^K du >\}] \le C(\gamma^2 + \gamma)T.
$$

The same bound holds if we replace J by a function dependent on time with uniform bound $||J_t||_\infty \leq 1$.

Proof. By the Feynman-Kac inequality, and conditioning as before, we have that the left hand side is bounded above by

$$TN \sup_{f \geq 0 : E_{\mu_{\rho^1,\rho^2}}[f]=1} \{E_{\mu_\rho}[\gamma X_i f]] - ND_{N,\rho}(\sqrt{f})\}$$

$$\leq TN \sup_f \sup_x \sup_{|\boldsymbol{y}| \leq \boldsymbol{R}} \sup_{\eta_2} \{E_{\nu_{K,\boldsymbol{y}}(\cdot|\eta_2)}[\gamma X_i(x)f]] - \frac{N^2}{K} D^{x,1}_{\nu_{K,\boldsymbol{y}}(\cdot|\eta_2)}(\sqrt{f})\}$$

We next apply the logarithmic Sobolev inequality, Theorem 4.5, followed by the entropy inequality to obtain that this is less than

$$T \frac{N^3}{K^3} \sup_{|\boldsymbol{y}| \leq \boldsymbol{R}} \sup_{\eta_2} \{\log E_{\nu_{K,\boldsymbol{y}}(\cdot|\eta_2)}[\exp\{\gamma K^3/N^2 X_i\}]\}.$$

Using Lemma 6.3 in the above along with $l\sqrt{N} = K$ proves the result. $\square$

We now turn to the calculation of the quadratic variation. For fixed α_1 and α_2 we have that $\alpha_1 < M^N_{t,1}, f_1 > + \alpha_2 < M^N_{t,2}, f_2 >$ has quadratic variation under P_N given by:

$$\sum_i \frac{\alpha_i^2}{2} \int_0^t N^{-1} \sum_{x \in \mathbb{T}^N} [c_i(\eta_s^N(x)) + c_i(\eta_s^N(x + \frac{1}{N}))][\nabla_N f(x)]^2 ds. \tag{6.11}$$

To show that in the limit these martingales match with (6.1) it is enough to prove the following proposition holds for $i = 1, 2$.

Proposition 6.6. *For a smooth function h in $C^1(\mathbb{T})$.*

$$E_{NEQ}\left[\left|\int_0^t < h, c_i(\eta_s^N) - \varphi_i(\rho_s) > ds\right|\right] \to 0.$$

Proof. Let $X_{s,i}$ denote the field $c_i(\eta_s^N) - \varphi_i(\rho_s)$ for $i = 1, 2$. Our first step is to replace $X_{s,i}$ with an average over boxes of size K. By summation by parts, the replacement adds a term of size $\frac{K}{N}$. Hence we need only worry about the inequality using the field averaged over boxes of size K : $X_{s,i}^K$. Fix a large R and consider

$$\begin{aligned} X_{s,i}^K(x) &= AV_{|x-y| \leq K/N} \left[c_i(\eta_s^N(x)) - \varphi_i(\rho_s(x))\right] \mathbb{I}[m^K(x) \leq R] \\ &\quad + AV_{|x-y| \leq K/N} \left[c_i(\eta_s^N(x)) - \varphi_i(\rho_s(x))\right] \mathbb{I}[m^K(x) > R] \\ &= X_{s,i}^{K,(a)}(x) + X_{s,i}^{K,(b)}(x) \end{aligned}$$

By Chebychev's inequality the second field, $X_i^{K,(b)}(x)$, is bounded by

$$E_{NEQ}[\int_0^t | < h, X_{i,s}^{K,(b)} > |ds] \leq \frac{C}{R}$$

for some constant C. By the entropy inequality, it follows from Lemma 6.5 and assumption (H1) that

$$E_{NEQ}[\int_0^t < h, X_{i,s}^{K,(a)} > ds] \leq \frac{C\beta}{N} + \frac{C}{\beta}.$$

The result follows if we first let N converge to ∞, then do the same for β, then R. $\square$

6.3. **Uniqueness of the Martingale Problem.** In this section we show that the martingale problem described by the generalized stochastic differential equation (1.6) has a unique solution. We state and prove the theorem only for 2 colours, however, the statement as well as the proof are valid for any number of colours.

Define for $\boldsymbol{f}$ in $\mathcal{H}_m^2$

$$< \boldsymbol{M}_t, \boldsymbol{f} > \;=\; < \boldsymbol{Y}_t, \boldsymbol{f} > - < \boldsymbol{Y}_0, \boldsymbol{f} > - \int_0^t < \frac{1}{2}\boldsymbol{D}_2(\boldsymbol{\rho})^*\boldsymbol{Y}_s, \triangle\boldsymbol{f} > ds. \quad (6.12)$$

We also define the operators $\mathcal{S}_{t,i} = \sqrt{\varphi_i(\boldsymbol{\rho}_t)}\nabla$, $\mathcal{A}_t^i = \sqrt{S'(\rho_t)\rho_{t,i}}\triangle$ and $\mathcal{P}_{t,u}^S$ the semigroup associated to the operator $\frac{1}{2}S(\rho_t)\triangle$.

Theorem 6.7. *Fix a positive integer $m \geq 2$. Let P be a probability measure on the space $\{(C[0,T], H_{-m}^2), \mathcal{F}\}$, where $\mathcal{F} = \cup_{t\geq 0}\mathcal{F}_t$, and $\mathcal{F}_t$ is the canonical filtration of the process $< \boldsymbol{Y}_t, \boldsymbol{f} >$. Assume that for any $\boldsymbol{f} \in C^\infty(\mathbb{T}) \times C^\infty(\mathbb{T})$ and $\boldsymbol{\alpha} \in \mathbb{R}^2$ the quantities $< \boldsymbol{M}_t, \boldsymbol{f} >$ defined in (6.12), and*

$$\{\boldsymbol{\alpha}\cdot < \boldsymbol{M}_t, \boldsymbol{f} >\}^2 - \sum_{i=1,2} \alpha_i^2 \int_0^t ||\mathcal{S}_{s,i}f_i||_2^2 ds \quad (6.13)$$

are $L^1(P)$ continuous martingales with respect to $\mathcal{F}_t$. Then, for any colour i, for all $0 \leq s \leq t$, f in $C^\infty(\mathbb{T})$, and subsets A of $\mathbb{R}$,

$$P[\left\{< Y_{t,i}, f > - \int_s^t < Y_u, \mathcal{A}_u^i \mathcal{P}^S(t,u)f > du\right\} \in A|\mathcal{F}_s]$$

$$= \int_A \frac{1}{\sqrt{2\pi \int_s^t ||\mathcal{S}_{u,i}\mathcal{P}_{t,u}^S f||_2^2 du}} \exp\left\{-\frac{(y- < Y_{i,s}, \mathcal{P}_{t,s}^S f >)^2}{2\int_s^t ||\mathcal{S}_{u,i}\mathcal{P}_{t,u}^S f||_2^2 du}\right\} dy, \quad (6.14)$$

P almost surely.

From this result we obtain uniqueness of the limiting measure P. From the above theorem we not only have the distribution of

$$< Y_{t,i}, f > - \int_s^t < Y_u, \mathcal{A}_u^i \mathcal{P}_{t,u}^S f >$$

for any colour i, but we also know that these quantities are independent for different i and different test functions f because of (6.13). If we choose the same f and sum over the colours we obtain

$$< Y_t, f > - \int_s^t < Y_u, \mathcal{A}_u \mathcal{P}_{t,u}^S f >,$$

where $\mathcal{A}_u$ is the sum over $\mathcal{A}_u^i$. This fact plus Theorem 5.8 which gives the full distribution of the process $< Y_t, f >$ is enough to obtain uniqueness of the pair $\{< Y_t^1, f >, < Y_t^2, h >\}$. A standard Markov argument gives the uniqueness of the finite dimensional distributions, which implies the uniqueness of the measure P.

The proof of the above theorem is similar to the proof of the uniqueness result for the colour-blind density fluctuation field as it appears in [KL] or [HS].

Proof. First of all notice that the martingale $< M_{t,i}, f >$ in (6.12) may also be written as

$$< M_{s,i}, f > + < Z_{t,s}^i, f > - \int_s^t < Y_u, \mathcal{A}_u^i f > du,$$

where $< Z_{t,s}^i, f >$ is used to denote

$$< Y_{t,i}, f > - < Y_{s,i}, f > - \int_s^t < Y_{u,i}, \frac{1}{2}S(\rho_u)\triangle f > du.$$

Itô's formula together with the fact that (6.12) and (6.13) are both martingales imply that for each fixed $s \geq 0$ and $f \in C^\infty(\mathbb{T})$, $X_{t,s}(f)$ defined by

$$X_{t,s}(f) = \exp\left\{i\left(< Z_{t,s}^i, f > - \int_s^t < Y_u, \mathcal{A}_u^i f > du\right) + \frac{1}{2}\int_s^t ||\mathcal{S}_{u,i}f||_2^2 du\right\}$$

is a martingale. Next take two fixed times $t_1 < t_2 \leq T$ and let $s_{n,j}$ denote a partition of the time interval defined by $s_{n,j} = t_1 + \frac{j}{n}(t_2 - t_1)$. Consider the quantity defined as

$$\Pi_{j=0}^{n-1} X_{s_{n,j+1}, s_{n,j}}(\mathcal{P}^S(T, s_{n,j})f).$$

By continuity of $< Y_s^i, \mathcal{P}^S(T, t)f >$ in $\{s, t\}$ we may let $n \to \infty$ to obtain that the above quantity converges a.s. and in $L^1(P)$ to $\frac{Z_{t_2}(f)}{Z_{t_2}(f)}$, where $Z_t(f)$ is equal to

$$\exp\left\{i < Y_{t,i}, \mathcal{P}_{T,t}^S f > - \int_0^t < Y_u, \mathcal{A}_u^i \mathcal{P}_{T,u}^S f > du + \frac{1}{2}\int_0^t ||\mathcal{S}_{u,i}\mathcal{P}_{T,u}f||_2^2 du\right\}.$$

Because the convergence above takes place also in $L^1(P)$ the martingale properties of $X_{t,s}(\mathcal{P}_{(T,s)}^S f)$ are passed onto $Z_t(f)$. The fact that $Z_t(f)$ is a martingale proves the theorem. $\square$

6.4. **Tightness.** Let P_N denote the probability measure on $D([0, T], \mathcal{H}_{-m}^2)$ induced by the fluctuation field $\mathbf{Y}_t^N$.

For a function $\mathbf{F} = \{F^1, F^2\}$ in $D([0, T], \mathcal{H}_{-m}^2)$, define the (uniform) modulus of continuity, $\omega_\delta(\mathbf{F})$, for a fixed $\delta > 0$:

$$\omega_\delta(\boldsymbol{F}) = \sup_{|s-t|\leq\delta, 0\leq s,t\leq T}\left\{||F_t^1 - F_s^1||_{-m} + ||F_t^2 - F_s^2||_{-m}\right\}.$$

As previously we shall simplify the supremum in the above notation to $\sup_{s,t}$. By Prohorov's theorem and Arzela-Ascoli it follows that to show the sequence P_N is tight we need to show that it satisfies two conditions:

(T1) $\lim_{B\to\infty} \limsup_{N\to\infty} P_N[\sup_{0\le t\le T} ||\boldsymbol{Y}_t||_{-m} > B] = 0$
(T2) $\lim_{\delta\to 0} \limsup_{N\to\infty} P_N[\omega_\delta(\boldsymbol{Y}) > \epsilon] = 0 \qquad \forall \epsilon > 0.$

Proposition 6.8. *The sequence of measures P_N is tight in $D([0,T], \mathcal{H}^2_{-4})$. Moreover, all limit points are concentrated on continuous paths.*

Using (6.2) we have that

$$< Y^N_{t,1}, f > = < Y^N_{0,1}, f > + \int_0^t < \tilde{Y}^N_{s,1}, \triangle_N f > ds + < M^N_{t,1}, f >,$$

where $M^N_{t,1}$ is a martingale. By assumption (F2) there is nothing to prove for the initial field. Also, $M^N_{t,1}$ has quadratic variation given in (6.11) with $\alpha_1 = 1$ and $\alpha_2 = 0$. This formula is very similar to the one in the single colour case, and hence, the proof will follow as before from the work we have already done in the two-colour Boltzmann-Gibbs principle. We may also handle each of $Y^N_{t,i}$ separately.

To show that $\boldsymbol{Y}^N_t$ satisfies condition (T2) it is enough to show

$(T2)_A$ $\lim_{\delta\to 0} \limsup_{N\to\infty} P_N[\sup_{s,t} \sup_{||f||_m\le 1}\{< \triangle_N f, \int_s^t \tilde{Y}^{N,i}_u du >\} > \frac{\epsilon}{4}]$
$(T2)_B$ $\lim_{\delta\to 0} \limsup_{N\to\infty} P_N[\sup_{s,t} ||M^N_{t,i} - M^N_{s,i}||_{-m} > \frac{\epsilon}{4}],$

for each of $i = 1, 2$.

We begin with $(T2)_B$, and fix $i = 1$. Clearly, the other case is the same. Because of this special "decoupling" that occurs, the proof is now almost identical to the colour-blind case, and we omit many of the details. The next three lemmas follow from the quadratic variation result in Proposition 6.6, as before; see the proof of Lemmas 5.11, 5.12 and 5.13.

Lemma 6.9 (Bounding the tails 1.)**.** *There exists a finite constant $C = C(\varphi)$ such that for every eigenfunction e_z of $\mathcal{H}_m$,*

$$\limsup_{N\to\infty} E_N[\sup_{0\le t\le T} |< M^N_{t,1}, e_z >|^2] \le CT\{1+ < \triangle_N e_z, \triangle_N e_z >\}$$

Lemma 6.10 (Bounding the tails 2.)**.** *For $m \ge 4$,*

$$\lim_{K\to\infty} \limsup_{N\to\infty} E_N[\sup_{0\le t\le T} \sum_{z\ge K}\{< M^N_{t,1}, e_z >\}^2 \gamma_z^{-m}] = 0.$$

Lemma 6.11. *For any f in $C^3(\mathbb{T})$*

$$\lim_{\delta\to 0} \limsup_{N\to\infty} P_N[\sup_{s,t}\{< M^N_{t,1} - M^N_{s,1}, f >\}^2 > \epsilon'] = 0.$$

These three lemma together prove that condition $(T2)_B$ is satisfied. We next turn to condition $(T2)_A$: yet again we split this field up in the sum of several terms:

$$
\begin{aligned}
\Psi_{1,t}^{N,1} &= \sqrt{N}\{c_1(\boldsymbol{\eta}_t^N(x)) - \bar{c}_{K,1}(x)\} \\
\Psi_{2,t}^{N,1} &= \sqrt{N}\{\bar{c}_{K,1}(\boldsymbol{\eta}_t^N(x)) - \varphi_1(\boldsymbol{m}_t^K(x))\}\mathbb{I}[m_t^K(x) \le R] \\
\Psi_{3,t}^{N,1} &= \sqrt{N}\partial_1\varphi_1(\boldsymbol{\rho}_t(x))\{m_{1,t}^K(x) - \rho_{1,t}(x)\}\mathbb{I}[m_t^K(x) \le R] \\
\Psi_{4,t}^{N,1} &= \sqrt{N}\partial_2\varphi_1(\boldsymbol{\rho}_t(x))\{m_{2,t}^K(x) - \rho_{2,t}(x)\}\mathbb{I}[m_t^K(x) \le R] \\
\Psi_{5,t}^{N,1} &= \sqrt{N}\Big[\varphi_1(\boldsymbol{m}_t^K(x)) - \varphi_1(\boldsymbol{\rho}_t(x)) \\
&\qquad - \sum_{i=1}^{2}\partial_i\varphi_1(\boldsymbol{\rho}_t(x))\{m_{i,t}^K(x) - \rho_{i,t}(x)\}\Big]\mathbb{I}[m_t^K(x) \le R] \\
\Psi_{6,t}^{N,1} &= -\sqrt{N}\varphi_1(\boldsymbol{\rho}_t(x))\mathbb{I}[m_t^K(x) > \tau] \\
\Psi_{7,t}^{N,1} &= \sqrt{N}\bar{c}_{K,1}(x)\mathbb{I}[m_t^K(x) > \tau].
\end{aligned}
\tag{6.15}
$$

We need to prove that

$$
\lim_{\delta \to 0}\limsup_{N \to \infty} P_N\left[\sup_{s,t}\ \sup_{\|f\|_m \le 1}\left|\int_s^t \Psi_{i,u}^{N,1}(\triangle_N f)du\right| > \frac{\epsilon}{28}\right],
\tag{6.16}
$$

for $i = 1, \ldots 7$. As before, the quantities $i = 2, 3, 4$ are more difficult.

For $i = 1, 5, 6, 7$ we have that (6.16) holds because we have already shown in the previous section

$$
\lim_{l \to 0}\lim_{N \to \infty} E_N\left[\sup_{\|f\|_m \le 1}\int_0^T\left|< \Psi_{i,u}^{N,1}, \triangle_N f >\right|du\right] = 0.
$$

That is, the result follows by summation by parts for $i = 1$, and by Theorem 6.4 and $|\Psi_{i,u}^N(x)| \le C\sum_{i=1,2}(m_i^K(x) - \rho^i(x))^2$ for the remaining cases.

It remains to show (6.16) for $i = 2, 3, 4$. By Propositions 5.15 and 5.16 this follows if we have the following bound for $i = 2, 3, 4$

$$
E_{EQ}]\left[\exp\{aN^{1/2}(t-s)^{-1/2} < \triangle_N f, \int_s^t \Psi_{i,u}^N du >\}\right] \le e^{Ca^2\{\|f\|_k^2+1\}},
$$

for a positive constant C and all large enough N. As before, to obtain convergence in $m \ge 4$ we need $k = 3$. Indeed this follows from Lemma 6.5 and the Sobolev inequality for $i = 2$. It is straightforward to check that the exact same bounds apply for $i = 3, 4$. The constant C depends on l and is valid for all $a > a_0$, that is, $C = C(l, a_0)$. This completes the proof of $(T2)_A$.

To complete the proof of tightness we hence need the bound (T1). Because of Lemma 6.9 and 6.10 we have that for $i = 1, 2$

$$\lim_{M \to \infty} \limsup_{N \to \infty} P_N \left[\sup_{0 \leq t \leq T} ||M_{t,i}^N||_{-m} > M \right] = 0.$$

We need only

$$\lim_{M \to \infty} \limsup_{N \to \infty} P_N \left[|| \int_0^T \tilde{Y}_{t,i}^N ||_{-m} > M \right] = 0,$$

again for $i = 1, 2$. However, this follows from Proposition 5.17 and Lemma 6.5. This completes the proof of tightness in the colour case.

References

[AB]   G. Ben Arous and M. Brunaud. Méthode de Laplace: étude variationelle des fluctuations des diffusions de type "champ moyen". *Stochastics and Stochastic Reports*, 31:79–144, 1990.

[Ald]   D.J. Aldous. Stopping times and tightness. *Annals of Probability*, 6(335-340), 1978.

[And]   E. Andjel. Invariant measures for the zero range processes. *Annals of Probability*, 10:525–547, 1982.

[CY]   C. C. Chang and H.T. Yau. Fluctuations of one dimensional Ginzburg-Landau models in nonequilibirium. *Communications in Mathematical Physics*, 69(145):209–234, 1992.

[DPP1] P. Dai Pra and G. Posta. Logarithmic Sobolev inequality for zero-range dynamics. *Annals of Probability*, 33(6):2355–2401, 2005.

[DPP2] P. Dai Pra and G. Posta. Logarithmic Sobolev inequality for zero-range dynamics: independence of the number of particles. *Electronic Journal of Probability*, 10:525–576, 2005.

[Eva]   M. R. Evans. Phase transitions in one-dimensional nonequilibrium systems. *Brazilian Journal of Physics*, 30(1), March 2000.

[FPV]   P.A. Ferrari, E. Presutti, and M.E. Vares. Non equilibrium fluctuations for a zero range process. *Ann. Inst. Henri Poincaré*, 24(2):237–268, 1988.

[GJLL] D. Gabrielli, G. Jona-Lasinio, and C. Landim. Onsager reciprocity relations without microscopic reversibility. *Physical Review Letters*, 77(7):1202–1205, 1996.

[GKL]   G. Gielis, A. Koukkous, and C. Landim. Equilibrium fluctuations for zero range processes in random environment. *Stoch. Proc. Appl.*, 77:187–205, 1998.

[GPV]   M.Z. Guo, G.C. Papanicolaou, and S.R.S. Varadhan. Nonlinear diffusion limit for a system with nearest neighbour interactions. *Commun. Math. Phys.*, 118:31–59, 1988.

[HS]   R.A. Holley and D.W. Stroock. Generalized Ornstein-Uhlenbeck processes and infinite particle branching brownian motions. *Publ. RIMS, Kyoto University*, 14:741–788, 1978.

[Jan]   H. Jankowski. Logarithmic Sobolev inequality for the inhomogeneous zero range process. Preprint., 2006.

[KL]   C. Kipnis and C. Landim. *Scaling Limits of Interacting Particle Systems*. Springer Verlag, Berlin, 1999.

[KV]   C. Kipnis and S.R.S. Varadhan. Central limit theorem for additive functionals of reversible markov processes and applications to simple exclusion. *Commun. Math. Phys.*, 106:1–19, 1986.

[LSV]   C. Landim, S. Sethuraman, and S.R.S. Varadhan. Spectral gap for zero range dynamics. *The Annals of Probability*, 24(4):1871–1902, 1996.

[Lu]   S.L. Lu. Equilibrium fluctuations of a one-dimensional nongradient Ginzburg-Landau model. *Annals of Probability*, 22:1252–1272, 1994.

[ML]   A. Martin-Löf. Limit theorems for the motions of poisson system of independent markovian particles at high density. *Z. Wahrsch. Verw. Gebiete*, 34:205–223, 1976.

[QRV]   J. Quastel, F. Rezakhanlou, and S.R.S. Varadhan. Large deviations for the symmetric simple exclusion process in dimensions $d \geq 3$. *Probability Theory and Related Fields*, 113:1–84, 1999.

[Qua]   J. Quastel. Diffusion of color in the simple exclusion process. *Comm. Pure Appl. Math*, 45:623–679, 1992.

[Rez]   F. Rezakhanlou. Propagation of chaos for symmetric simple exclusions. *Commun. Pure Appl. Math.*, 117:943–957, 1994.

[RM]   R. Richtmyer and D. Morton. *Difference Methods for Initial-Value Problems*. John Wiley, New York, 2nd edition, 1967.

[Ros]   H. Rost. *Dynamics and Processes*, volume 1031 of *Lecture Notes in Mathematics*, pages 97–107. Springer-Verlag, Berlin, 1983.

[Spi]   F. Spitzer. Interaction of Markov processes. *Advances in Mathematics*, 5:142–154, 1970.

[ST]   T. Shiga and H. Tanaka. Central limit theorem for a system of Markovian particles with mean field interactions. *Zeitschrift für Wahrscheinlichkeitstheorie und vervandte Gebiete*, 69:439–459, 1985.

[Szn1]   A.S. Sznitman. Nonlinear reflecting diffusion process, and the propagation of chaos and fluctuations associated. *Journal of Functional Analysis*, 36:311–36, 1984.

[Szn2]   A.S. Sznitman. *A Fluctuation result for nonlinear diffusions*, pages 145–160. Infinite Dimensional Analysis and Stochastic Processes. Pitman, Boston, 1985.

[Szn3]   A.S. Sznitman. *Topics in Propagation of Chaos*, volume 1464 of *Lecture Notes in Math*, pages 165–251. 1989.

[Tan]   H. Tanaka. *Stochastic Analysis*, volume 32, pages 469–488. North Holland Math. Library, 1984.

[Var]   S.R.S. Varadhan. *Stochastic Processes*. Courant Institute of Mathematical Science, New York University, New York, 1968.

DEPARTMENT OF STATISTICS, UNIVERSITY OF WASHINGTON
E-MAIL: hanna@stat.washington.edu